Anticipating Random Periodic Solutions–I. SDEs with Multiplicative Linear Noise

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Abstract

In this paper, we study the existence of random periodic solutions for semilinear stochastic differential equations. We identify them as solutions of coupled forward-backward infinite horizon stochastic integral equations (IHSIEs), using the "substitution theorem" of stochastic differential equations with anticipating initial conditions. In general, random periodic solutions and the solutions of IHSIEs, are anticipating. For the linear noise case, with the help of the exponential dichotomy given in the multiplicative ergodic theorem, we can identify them as the solutions of infinite horizon random integral equations (IHSIEs). We then solve a localised forward-backward IHRIE in $C(\mathbb{R}, L^2_{loc}(\Omega))$ using an argument of truncations, the Malliavin calculus, the relative compactness of Wiener-Sobolev spaces in $C([0,T], L^2(\Omega))$ and Schauder's fixed point theorem. We finally measurably glue the local solutions together to obtain a global solution in $C(\mathbb{R}, L^2(\Omega))$. Thus we obtain the existence of a random periodic solution and a periodic measure.

Keywords: random periodic solutions, periodic measures, semilinear stochastic differential equations, relative compactness, Malliavin derivative, infinite horizon stochastic integral equations, exponential dichotomy.

1 Introduction

Many real world phenomena having the nature of mixing periodicity and randomness, e.g., change of temperatures on earth and seasonal economic data ([17]). Physicists have attempted to study random perturbations to periodic solutions for some time. They used first linear approximations or asymptotic expansions in small noise regime, e.g. see [29],[31]. The approach in [31] was to seek $Y(t+\tau,\omega)$ returning to a neighbourhood of $Y(t,\omega)$ for each noise realisation, where $\tau > 0$ is a fixed number. This reveals certain information about the "periodicity" under small noise perturbations. Efforts were also made in the mathematics community to seek a periodic solution Y such that $Y(0,\omega) = Y(\tau,\omega) = Y(2\tau,\omega) = \cdots$ ([7],[28]). However, this kind of strict periodicity exists only in very special situations in stochastic contexts. In general,

random perturbations to a periodic solution break the strict periodicity immediately, similar to the case that random perturbations break fixed points. The concept of stationary solutions is the corresponding notion of fixed points in the stochastic counterpart and has been subject to intensive study in mathematics literature ([11],[21],[27],[32]). One of the obstacles to make a systematic progress in the study of the random periodicity was the lack of a rigorous mathematical definition in a great generality and appropriate mathematical tools.

Recently random periodic solutions has been defined with a help of an observation that they may be regarded as stationary solutions on sequences of fixed discrete times of equal length of one period. The existence was studied for cocycles in [33] using a qualitative method, and for stochastic semiflows in [13],[14] using some analysis tools. There have been some more recent results including [8] on random attractors of the stochastic TJ model in climate dynamics, [30] on bifurcations of stochastic reaction diffusion equations and [5] on stochastic lattice systems. It was recently proved in [15] that random periodic solutions and periodic measures are "equivalent" in the following sense. A random periodic solution gives rise to a periodic measure. Conversely from a periodic measure one can construct a random periodic process on an enlarged probability space, of which the law is the periodic measure. It was then proved that the strong law of large numbers (SLLN) holds for periodic measures and corresponding random periodic processes thus it gives a statistical description. There are numerous physically relevant stochastic differential equations satisfying the conditions in this paper so they have random periodic solutions (Theorems 4.9, 4.11, 5.2).

First, recall the definition of the random periodic solutions for stochastic semi-flows given in [13]. Let X be a separable Banach space. Denote by $(\Omega, \mathcal{F}, P, (\theta(s))_{s\in\mathbb{R}})$ a metric dynamical system, where $\theta(s) : \Omega \to \Omega$ is assumed to be P-preserving and measurably invertible for all $s \in \mathbb{R}$, $\Delta := \{(t,s) \in \mathbb{R}^2, s \leq t\}$. Consider a stochastic semi-flow $u : \Delta \times \Omega \times X \to X$, which satisfies the following standard condition

$$u(t,r,\omega) = u(t,s,\omega) \circ u(s,r,\omega), \quad \text{for all } r \leq s \leq t,\ r,s,t \in \mathbb{R},\ \text{for } a.e.\ \omega \in \Omega. \tag{1.1}$$

We do not assume the map $u(t,s,\omega) : X \to X$ to be invertible for $(t,s) \in \Delta,\ \omega \in \Omega$ in the following definition.

Definition 1.1. *A random periodic path of period τ of the semi-flow $u : \Delta \times \Omega \times X \to X$ is an $\mathcal{F}$-measurable map $Y : \mathbb{R} \times \Omega \to \mathbb{R}^d$ such that*

$$u(t,s,Y(s,\omega),\omega) = Y(t,\omega),\ Y(s+\tau,\omega) = Y(s,\theta_\tau\omega), \quad t,s \in \mathbb{R},\ t \geq s, \quad a.e. \omega \in \Omega. \tag{1.2}$$

In this paper we consider the following stochastic differential equation with multiplicative noise since the influence of noise in many cases depends on the intensity of the solution

$$\begin{cases} du(t) = Au(t)\,dt + F(t,u(t))\,dt + \sum_{k=1}^{M} B_k(t,u(t)) \circ dW_t^k, & t \geq s \\ u(s) = x \in \mathbb{R}^d, \end{cases} \tag{1.3}$$

where $W_t = \{W_t^k\ 1 \leq k \leq M, t \in \mathbb{R}\}$ is an M-dimensional mutually-independent standard Brownian motion under the canonical probability space $(\Omega, \mathcal{F}, (\mathcal{F}^t)_{t\in\mathbb{R}}, \mathbb{P})$. Assume there exists

a constant $\tau > 0$ such that $F(t+\tau, u) = F(t, u)$, $B_k(t+\tau, u) = B_k(t, u)$. Such kind of SDEs is called τ-periodic. Suppose that the continuous functions $F(t, u)$ and $B_k(t, u)$ are Lipschitz continuous in u so that the initial value problem (1.3) has a unique solution (see [18]). It is easy to see that the Lipschitz condition is always satisfied under the conditions given in the following sections.

Note the stochastic integral is of the Stratonovich type. Recall the connection of the Itô and Stratonovich integrals

$$B_k(t, u(t)) \circ dW_t^k = B_k(t, u(t))dW_t^k + \frac{1}{2}\nabla_x B_k(t, u(t))B_k(t, u(t))dt. \tag{1.4}$$

Random periodic solutions of τ-periodic semilinear stochastic differential equations with additive noise were studied in [13] ($B_k(t, u) = B_k(t)$). The existence was obtained under conditions that $||F||_\infty < \infty, ||\nabla F||_\infty < \infty, ||B_k||_\infty < \infty$, $B_k(t)$ is Hölder continuous and the matrix A is hyperbolic. There is no requirement about F being monotone or the Lipschitz constant being controlled by the spectrum of A. In fact the system is non-dissipative as it is contracting in certain directions and expanding in certain other directions. So the random periodic solution, if exists, is not stable or completely unstable. The pull-back method does not work. In fact the random periodic solution is an anticipating stochastic process, which depends on the whole path of the noise.

If the stochastic system is dissipative, we can use other methods such as pull-back to study the problem, where there is no anticipating issue at all ([33]). In this paper, we consider non-dissipative systems with the $d \times d$ matrix A being assumed to be hyperbolic. Similar to the case with additive noise, if this equation has a random periodic solution Y, it is also anticipating. However the nonadaptedness causes a real difficulty in the analysis of $\int_0^t B_k(s, Y(s)) \circ dW_s^k$, though this term is still well defined through (1.4). The first integral on the right hand side is of the Skorohod type, of which the L^2-norm is given by the Skorohod isometry ([25])

$$\begin{aligned} & || \int_0^t B_k(s, Y(s))dW_s^k||^2_{L^2(\Omega)} \qquad (1.5)\\ = & \int_0^t \|B_k(s, Y(s))\|^2_{L^2(\Omega)}ds + \int_0^t \int_0^t (\nabla_x B_k(s, Y(s))\mathcal{D}^k_{r_1}Y(s))^T \nabla_x B_k(s, Y(s))\mathcal{D}^k_{r_2}Y(s)dr_1dr_2, \end{aligned}$$

where A^T is the transpose of A and $\mathcal{D}Y$ the Malliavin derivative of Y.

For a finite time SDEs with a given anticipating initial condition, Nualart proved a substitution theorem in [25]. With the help of this result, we can prove that the random periodic solution is identified as a solution of the following IHSIEs

$$\begin{aligned} Y(t) = & \int_{-\infty}^t \mathrm{e}^{A(t-r)}P^- F(r, Y(r))dr - \int_t^{+\infty} \mathrm{e}^{A(t-r)}P^+ F(r, Y(r))dr \qquad (1.6)\\ & + \sum_{k=1}^M \int_{-\infty}^t \mathrm{e}^{A(t-r)}P^- B_k(r, Y(r)) \circ dW_r^k - \sum_{k=1}^M \int_t^{+\infty} \mathrm{e}^{A(t-r)}P^+ B_k(r, Y(r)) \circ dW_r^k. \end{aligned}$$

Here P^- is the projection from $\mathbb{R}^d$ to the subspace spanned by the eigenvectors corresponding to the eigenvalue of A with negative real parts and $P^+ = I - P^-$. Thus our problem is reduced to solve (1.6). Note here Eqn. (1.6) can be represented as an initial value problem (1.3) with initial value $u(s) = Y(s)$ as for $t \geq s$, from (1.6),

$$\begin{aligned} Y(t) &= \mathrm{e}^{A(t-s)}\int_{-\infty}^{s} \mathrm{e}^{A(s-r)}P^-F(r,Y(r))dr + \int_s^t \mathrm{e}^{A(t-r)}P^-F(r,Y(r))dr \\ &\quad + \mathrm{e}^{A(t-s)}\sum_{k=1}^{M}\int_{-\infty}^{s} \mathrm{e}^{A(s-r)}P^-B_k(r,Y(r))\circ dW_r^k + \int_s^t \mathrm{e}^{A(t-r)}P^-B_k(r,Y(r))\circ dW_r^k \\ &\quad - \mathrm{e}^{A(t-s)}\int_s^{+\infty} \mathrm{e}^{A(s-r)}P^+F(r,Y(r))dr + \int_s^t \mathrm{e}^{A(t-r)}P^+F(r,Y(r))dr \\ &\quad - \mathrm{e}^{A(t-s)}\sum_{k=1}^{M}\int_s^{+\infty} \mathrm{e}^{A(s-r)}P^+B_k(r,Y(r))\circ dW_r^k + \sum_{k=1}^{M}\int_s^t \mathrm{e}^{A(t-r)}P^+B_k(r,Y(r))\circ dW_r^k \\ &= \mathrm{e}^{A(t-s)}Y(s) + \int_s^t \mathrm{e}^{A(t-r)}F(r,Y(r))dr + \sum_{k=1}^{M}\int_s^t \mathrm{e}^{A(t-r)}B_k(r,Y(r))\circ dW_r^k. \end{aligned} \tag{1.7}$$

This would be exactly the type of SDEs considered in [25] with a given anticipating initial value $Y(s)$ at time s if it were known. However, in our context, $Y(s)$ is not known, but part of the solution of Eqn. (1.6) when $t = s$.

It turns out that the anticipating issue causes some real difficulties in solving (1.6), even in the linear noise case, due to the fact that the L^2 norm of $\int_0^t B_k(s, Y(s)) \circ dW^k(s)$ involves the Malliavin derivative of Y in (1.5). In this paper, we only consider the linear noise case. We use the stochastic linear evolution operator and subsequently identify random periodic solutions as the solutions of forward-backward coupled infinite horizon random integral equations (IHRIEs) with the help of the exponential dichotomy given in Oseledets' multiplicative ergodic theorem (MET).

We cannot solve the IHRIEs pathwise though the equations are given in a pathwise manner. The major flaw of a pathwise approach is the lack of the measurability to their solutions. Thus we seek a solution in $C((-\infty,+\infty), L^2(\Omega))$. Relative compactness is key in this analysis. However, pointwise (fix ω) relative compactness theorem such as Arzelà-Ascoli Lemma is not enough. Another difficulty is the pathwise stability of $\Phi(t, \theta_s\omega)P^-$ in the MET and the stability in $L^2(\Omega)$ are different. For example, as $t \to \infty$, $e^{-\frac{1}{2}t+W_t} \to 0$ a.s., but $\mathbb{E}\left(e^{-\frac{1}{2}t+W_t}\right)^2 \to \infty$. To overcome this difficulty, we construct a sequence of localised IHRIEs. We then use the relative compactness of Wiener-Sobolev spaces in $C([0,T], L^2(\Omega))$ and Schauder's fixed point theorem to solve the equation in $C(\mathbb{R}, L^2_{loc}(\Omega))$. We finally measurably glue the local solutions together to obtain a global solution in $C(\mathbb{R}, L^2(\Omega))$. It is interesting to note that the local solution may not converge to the global solution.

With the existence of random periodic solution, we can construct a periodic measure μ_s of the skew product on $(\Omega \times R^d, \mathcal{F} \bigotimes \mathcal{B}(R^d))$ according to [15]. The factorisation of μ_s which is given by $(\mu_s)_\omega = \delta_{Y(s,\theta(-s)\omega)}$ is anticipating.

We would like to point out that anticipating equilibrium processes exist in reality. For example, an anticipating stationary solution was found in the transition state problem in chemical reaction processes (c.f. [4]).

2 Preliminaries and the equivalence of random periodic solutions for hyperbolic systems and the coupled forward-backward IHSIEs

Consider $\Omega = C_0(\mathbb{R}, \mathbb{R}^M) := \{\omega \in C(\mathbb{R}, \mathbb{R}^M) : \omega(0) = 0\}$, $W_t(\omega) := \omega(t)$, and $\mathcal{F}^t := \vee_{s\leq t}\mathcal{F}_s^t$ with $\mathcal{F}_s^t := \sigma(W_u - W_v, s \leq v \leq u \leq t)$. Besides, we define a shift that leaves Ω invariant by

$$\theta_t : \Omega \to \Omega, \quad \theta_s\omega(t) = \omega(t+s) - \omega(s), \ s, t \in \mathbb{R},$$

and thus the shift θ is P-measure preserving.

First we briefly recall the Skorohod integral, Stratonovich intergal and prove Malliavin derivative's norm preserving property under the measure preserving operator θ. We only need to consider 1-dimensional Brownian motion W on $\mathbb{R}$. The multidimensional case can be dealt with similarly.

Denote by $\hat{L}^2(\mathbb{R}^m)$ the set of symmetric functions in $L^2(\mathbb{R}^m)$. Define for $f \in \hat{L}^2(\mathbb{R}^m)$,

$$I_m(f) = \int_{\mathbb{R}^m} f(t_1, \ldots, t_m) dW_{t_1} \ldots dW_{t_m}.$$

It is well known that

$$\mathbb{E}[I_m(f)I_n(g)] = \begin{cases} 0, & if \ n \neq m, \\ m! < f, g >_{L^2(\mathbb{R}^m)}, & if \ n = m. \end{cases} \tag{2.1}$$

Definition 2.1. (Skorohod integral [24]) Suppose that $v(t, \omega)$ is a stochastic process such that $v(t, \cdot)$ is $\mathcal{F}$-measurable and square-integrable for all $t \in \mathbb{R}$. Thus it has the following Wiener-Itô chaos expansion

$$v(t) = \sum_{m=0}^{\infty} I_m(f_m(\cdot, t)),$$

with $f_m(\cdot, t) \in \hat{L}^2(\mathbb{R}^m)$ for each $t \in \mathbb{R}$. Then the Skorohod integral is defined as

$$\delta(v) := \int_{\mathbb{R}} v(t, \omega)\delta W_t := \sum_{m=0}^{\infty} I_{m+1}(\tilde{f}_m), \tag{2.2}$$

where $\tilde{f}_m \in \hat{L}^2(\mathbb{R}^{m+1})$ is the symmetrization of $f_m(t_1, \cdots, t_m, t)$ as a function of all $m+1$ variables. We say u is Skorohod-integrable and write $u \in Dom(\delta)$ if the series in (2.2) converges in $L^2(\Omega)$.

Another kind of integral is defined in the probability sense (1-dimensional case):

Definition 2.2. (Stratonovich integral [25]) A measurable process $v(t,\omega)$ such that $\int_{\mathbb{R}} |v_t| dt < \infty$ a.s. is Stratonovich integrable if the familiy S^π

$$S^\pi := \int_{\mathbb{R}} v_t W_t^\pi dt,$$

where

$$W_t^\pi = \sum_{i=0}^{n-1} \frac{W_{t_{i+1}} - W_{t_i}}{t_{i+1} - t_i} \chi_{(t_i, t_{i+1}]}(t),$$

converges in probability as $|\pi| \to 0$ and in this case the limit is called the Stratonovich integral, denoted by $\int_{\mathbb{R}} v_t \circ dW_t$.

Note that the relation (1.4) holds between the Stratonovich integral and the Skorohod integral (c.f. [25]), i.e.,

$$\int_{\mathbb{R}} B(s,u(s)) \circ dW_s = \int_{\mathbb{R}} B(s,u(s)) \delta W_s + \frac{1}{2} \int_{\mathbb{R}} \nabla_x B(s,u(s)) B(s,u(s)) ds, \tag{2.3}$$

where $B \in \mathcal{L}(\mathbb{R}^d)$.

Let $\mathcal{S}$ denote the class of smooth and cylindrical random variables of the form

$$G = f(W(h_1), \cdots, W(h_n)),$$

where $f \in C_p^\infty(\mathbb{R}^n)$, i.e., f and all its partial derivatives have polynomial growth order, and $W(h_i) = \int_{\mathbb{R}} h_i(s) dW_s$, $h_1, \cdots, h_n \in L^2(\mathbb{R})$, and $n \geq 1$. The derivative of G is the $L^2(\mathbb{R})$-valued random variable given by

$$\mathcal{D}_r G = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n)) h_i(r).$$

Then denote by $\mathcal{D}^{1,p}$ the domain of $\mathcal{D}$ in $L^p(\Omega)$, i.e. $\mathcal{D}^{1,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$||G||_{1,p} = \left(\mathbb{E}|G|^p + \mathbb{E}||\mathcal{D}.G||^p_{L^2(\mathbb{R})} \right)^{\frac{1}{p}}.$$

The following simple result is about the $\mathcal{D}^{1,2}$-norm-preserving property under the measure preserving operator. It will play a crucial role in the subsequent argument. This property is not normally true for Malliavin derivatives, but it is here due to the fact that the time interval is the whole real line $\mathbb{R}$.

Lemma 2.3. (Norm preserving in $\mathcal{D}^{1,2}$) Suppose $G(\cdot) \in \mathcal{D}^{1,2}$, then for all $h \in \mathbb{R}$, $G(\theta_h \cdot) \in \mathcal{D}^{1,2}$, and

$$\|G(\theta_h \cdot)\|_{1,2} = \|G(\cdot)\|_{1,2},$$

where $\theta_h : \Omega \to \Omega$, $h \in \mathbb{R}$ is the measure preserving measurable dynamical system on $(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. First by the measure-preserving property, it is easy to see that

$$\mathbb{E}|G(\theta_h\cdot)|^2 = \mathbb{E}|G(\cdot)|^2.$$

By Wiener-Itô's chaos decomposition (c.f. [24]), $G(\cdot) \in \mathcal{D}_{1,2}$ can be written as

$$G(\omega) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_k(t_1, \cdots, t_k) dW_{t_1} \cdots dW_{t_k},$$

where $f_k(-)$ is a symmetric element in $\hat{L}^2(\mathbb{R}^k)$, and

$$G(\theta_h\omega) \;=\; \sum_{k=0}^{\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_k(t_1 - h, \cdots, t_k - h) dW_{t_1} \cdots dW_{t_k}.$$

The corresponding Malliavin derivatives can be derived through Wiener-Itô chaos decomposition,

$$\mathcal{D}_r G(\omega) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_k(t_1, \cdots, t_{k-1}, r) dW_{t_1} \cdots dW_{t_{k-1}},$$

and

$$\mathcal{D}_r G(\theta_h\omega) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_k(t_1 - h, \cdots, t_{k-1} - h, r - h) dW_{t_1} \cdots dW_{t_{k-1}}.$$

Therefore

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{R}} \|\mathcal{D}_r G(\theta_h\cdot)\|^2 dr &= \sum_{k=0}^{\infty}(k-1)! \int_{\mathbb{R}} \|f_k(t_1 - h, \cdots, t_{k-1} - h, r - h)\|^2_{L^2(\mathbb{R}^{k-1})} dr \\
&= \sum_{k=0}^{\infty}(k-1)! \int_{\mathbb{R}^k} |f_k(t_1, \cdots, t_{k-1}, r)|^2 dt_1 \cdots dt_{k-1} dr \\
&= \mathbb{E}\int_{\mathbb{R}} \|\mathcal{D}_r G(\cdot)\|^2 dr.
\end{aligned}$$

The claim is asserted. □

Note that when $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical probability space associated with an M-dimensional Brownian motion $\{W_t^j, t \in \mathbb{R}, 1 \le j \le M\}$, $\mathcal{D}G$ of a random variable $G \in \mathcal{D}^{1,2}$ will be an M-dimensional process denoted by $\{\mathcal{D}_r^j G, r \in \mathbb{R}, 1 \le j \le M\}$. For example

$$\mathcal{D}_r^j W_t^k = \delta_{k,j}\chi_{(-\infty,t]}(r).$$

Here and throughout the paper $\chi_{\cdot}(\cdot)$ always represents an indicator function. Denote the solution of the initial value problem (1.3) by $u(t,s,\omega)x$, $t \ge s$, and

$$u(t,s,x) = e^{A(t-s)}x + \int_s^t e^{A(t-r)} F(r, u(r,s,x)) dr + \sum_{k=1}^{M} \int_s^t e^{A(t-r)} B_k(r, u(r,s,x)) dW_r^k.$$

It is easy to see that u satisfies (1.1). Moreover, by Proposition 2.3.22 in [2] and periodicity of F and B_k, similar to the proof cocycle case in Theorem 2.3.26 in [2], we have that

$$\begin{aligned}
& u(t+\tau, s+\tau, x) \\
= \;& e^{A(t-s)}x + \int_{s+\tau}^{t+\tau} e^{A(t+\tau-r)} F(r, u(r, s+\tau, x))dr + \sum_{k=1}^{M} \int_{s+\tau}^{t+\tau} e^{A(t+\tau-r)} B_k(r, u(r, s+\tau, x))dW_r^k \\
= \;& e^{A(t-s)}x + \int_{s}^{t} e^{A(t-r)} F(r+\tau, u(r+\tau, s+\tau, x))dr \\
& + \sum_{k=1}^{M} \int_{s}^{t} e^{A(t-r)} B_k(r+\tau, u(r+\tau, s+\tau, x))d\widetilde{W}_r^k \\
= \;& e^{A(t-s)}x + \int_{s}^{t} e^{A(t-r)} F(r, u(r+\tau, s+\tau, x))dr + \sum_{k=1}^{M} \int_{s}^{t} e^{A(t-r)} B_k(r, u(r+\tau, s+\tau, x))d\widetilde{W}_r^k,
\end{aligned}$$

where $\widetilde{W}_r := (\theta_\tau \omega)(r) = W_{r+\tau} - W_\tau$. On the other hand,

$$\theta_\tau u(t, s, x) \;=\; e^{A(t-s)}x + \int_{s}^{t} e^{A(t-r)} F(r, \theta_\tau u(r, s, x))dr + \sum_{k=1}^{M} \int_{s}^{t} e^{A(t-r)} B_k(r, \theta_\tau u(r, s, x))d\widetilde{W}_r^k.$$

By the pathwise uniqueness of the solution of (1.3), we have that

$$u(t+\tau, s+\tau, \omega) = u(t, s, \theta_\tau \omega), \text{ for all } t \geq s, t, s \in \mathbb{R}, \ a.s. \tag{2.4}$$

Note that $u(t, s, \omega)$ is also well-defined when $t \leq s$ and satisfies (c.f. [20])

$$\begin{cases} du(t) = -[Au(t) + F(t, u(t)) + \sum_{k=1}^{M} \nabla_x B_k(t, u(t)) B_k(t, u(t))]dt - \sum_{k=1}^{M} B_k(t, u(t))dW_t^k, \\ u(s) = x \in \mathbb{R}^d. \end{cases} \tag{2.5}$$

This means that the stochastic semi-flow is invertible a.s. Although this is not essential to make our method working, it helps to derive the IHSIEs. In the case of SPDEs, this property does not hold. In [16], applying the unstable manifold theorem ([23]), we can still deduce the IHSIEs in the infinite dimensional case.

The following substitution theorem for anticipating stochastic differential equations in [25] will play an important role in the development of the connection between the IHSIE and random periodic solutions.

Lemma 2.4. *Consider the following stochastic differential equation on* $[0, T]$, $T > 0$

$$X_{t,s} \;=\; X_0 + \sum_{i=1}^{M} \int_{s}^{t} \sigma_i(\hat{s}, X_{\hat{s},s})dW_{\hat{s}}^i + \int_{s}^{t} \beta(\hat{s}, X_{\hat{s},s})d\hat{s}, \quad t \geq s, \tag{2.6}$$

where $\sigma_i \in C^3(\mathbb{R}^{d+1})$, $0 \leq i \leq M$, *and* $\beta \in C^3(\mathbb{R}^{d+1})$ *have bounded partial derivatives of first order. Then for any random vector* X_0, *the process* $X = \{\varphi_{t,s}(X_0), t \in [0, T]\}$ *satisfies the*

anticipating SDEs (2.6), where $\{\varphi_{t,s}(x), t \in [0,T]\}$ *is the stochastic flow defined by:*

$$\varphi_{t,s}(x) \;=\; x + \sum_{i=1}^{M} \int_s^t \sigma_i(\hat{s}, \varphi_{\hat{s},s}(x)) dW_{\hat{s}}^i + \int_s^t \beta(\hat{s}, \varphi_{\hat{s},s}(x)) d\hat{s}, \quad t \geq s. \tag{2.7}$$

Besides, if $\sigma_i, 1 \leq i \leq M$, *and* β *are of class* C^4, *and* $X_0 \in \mathcal{D}^{1,p}(\mathbb{R}^d)$ *for some* $p > 4$, *then the process* X *is the unique solution to (2.6) in* $L^2([0,T], \mathcal{D}^{1,4}(\mathbb{R}^d))$ *and is continuous in* t *almost surely.*

The equations considered in [25] is time independent case. But time dependent case can be easily deduced to time independent case by considering $\tilde{X}_t = \begin{bmatrix} t+s \\ X_{t+s,s} \end{bmatrix}$.

Now we consider the general τ-periodic SDEs with multiplicative noise (1.3).

Theorem 2.5. (Equivalence theorem) Let $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $B_k : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be of class C^3, with the Jacobians $\nabla F(t,\cdot)$ and $\nabla B_k(t,\cdot)$ globally bounded, for $1 \leq k \leq M$, i.e. $\sup_{t\in\mathbb{R}, x\in\mathbb{R}^d} ||\nabla F(t,x)||_{\mathcal{L}(\mathbb{R}^d)} + \sup_{t\in\mathbb{R}, x\in\mathbb{R}^d} ||\nabla B_k(t,x)||_{\mathcal{L}(\mathbb{R}^d)} < \infty$. Assume $F(t,u) = F(t+\tau, u)$ and $B_k(t,u) = B_k(t+\tau, u)$ for some fixed $\tau > 0$. Then a tempered $Y \in L^2([0,\tau), \mathcal{D}^{1,p})$ for some $p > 4$ such that $Y(t+\tau, \omega) = Y(t, \theta_\tau \omega)$ for any $t \in \mathbb{R}$ $\mathbb{P}$-a.s. is a random periodic solution of Eqn. (1.3) if and only if Y satisfies Eqn. (1.6).

Proof. If Eqn. (1.6) has a solution $Y \in \mathcal{D}^{1,p}(L^2(\mathbb{R}, \mathbb{R}^d))$ for some $p > 4$, then it also satisfies (1.7). Thanks to Nualart's substitution theorem (Lemma 2.4), which guarantees the uniqueness of solution to (1.3) with anticipating initial value $Y(s,\omega)$, we have

$$u(t,s,x,\omega)\Big|_{x=Y(s,\omega)} = u(t,s,Y(s,\omega),\omega) = Y(t,\omega).$$

Conversely, assume Eqn. (1.3) has a random periodic solution which is tempered from above. First note for any non-negative integer l, we then apply the substitution theorem again to obtain

$$\begin{aligned} Y(t,\omega) \;&=\; u(t \pm l\tau, t, Y(t, \theta_{\mp l\tau}\omega), \theta_{\mp l\tau}\omega) \\ &=\; \mathrm{e}^{\mp Al\tau} Y(t, \theta_{\mp l\tau}\omega) + \int_t^{t\pm l\tau} \mathrm{e}^{A(t\pm l\tau - \hat{s})} F(\hat{s}, u(\hat{s}, t, Y(t, \theta_{\mp l\tau}\omega), \theta_{\mp l\tau}\omega)) d\hat{s} \\ &\quad + \sum_{k=1}^{M} \int_t^{t\pm l\tau} \mathrm{e}^{A(t\pm l\tau - \hat{s})} B_k(\hat{s}, u(\hat{s}, t, Y(t, \theta_{\mp l\tau}\omega), \theta_{\mp l\tau}\omega)) \circ dW_{\hat{s}\mp l\tau}^k. \end{aligned}$$

Therefore,

$$\begin{aligned} P^- Y(t,\omega) \;&=\; P^- u(t + l\tau, t, Y(t, \theta_{-l\tau}\omega), \theta_{-l\tau}\omega) \\ &=\; \mathrm{e}^{Al\tau} P^- Y(t, \theta_{-l\tau}\omega) + \int_{t-l\tau}^{t} \mathrm{e}^{A(t-\hat{s})} P^- F(\hat{s}, Y(\hat{s},\omega)) d\hat{s} \\ &\quad + \sum_{k=1}^{M} \int_{t-l\tau}^{t} \mathrm{e}^{A(t-\hat{s})} P^- B_k(\hat{s}, Y(\hat{s},\omega)) \circ dW_{\hat{s}}^k \end{aligned}$$

$$\to \int_{-\infty}^{t} \mathrm{e}^{A(t-\hat{s})} P^- F(\hat{s}, Y(\hat{s},\omega)) d\hat{s} + \sum_{k=1}^{M} \int_{-\infty}^{t} \mathrm{e}^{A(t-\hat{s})} P^- B_k(\hat{s}, Y(\hat{s},\omega)) \circ dW_{\hat{s}}^k$$

in L^2-norm as $l \to +\infty$. The last convergence can be demonstrated by the Skorohod isometry and (2.3). Indeed by the linear growth of B and boundedness of its gradients, for each k

$$\begin{aligned} & \mathbb{E}\left[\left|\int_{-\infty}^{t-l\tau} \mathrm{e}^{A(t-\hat{s})} P^- B_k(\hat{s}, Y(\hat{s},\omega)) \delta W_{\hat{s}}^k\right|^2\right] \\ \le\ & 2C_1 \int_{-\infty}^{t-l\tau} e^{2\mu_{m+1}(t-\hat{s})} (1 + |Y(\hat{s})|^2) d\hat{s} \\ & + C\|\nabla B_k\|_\infty^2 \mathbb{E} \int_{-\infty}^{t-l\tau} e^{2\mu_{m+1}(t-\hat{s})} \int_{-\infty}^{t-l\tau} \|\mathcal{D}_r Y(\hat{s},\omega)\|^2 dr d\hat{s}. \end{aligned}$$

Let us consider the second term on the right hand side only, as the first term can be dealt with analogously,

$$\begin{aligned} & \mathbb{E} \int_{-\infty}^{t-l\tau} e^{2\mu_{m+1}(t-\hat{s})} \int_{-\infty}^{t-l\tau} \|\mathcal{D}_r Y(\hat{s},\omega)\|^2 dr d\hat{s} \\ \le\ & e^{2\mu_{m+1} l\tau} \int_{-\infty}^{t} e^{2\mu_{m+1}(t-\hat{s})} \mathbb{E} \int_{-\infty}^{\infty} \|\mathcal{D}_r Y(\hat{s},\omega)\|^2 dr d\hat{s} \\ \le\ & e^{2\mu_{m+1} l\tau} \sum_{i=-1}^{\infty} e^{2\mu_{m+1} i\tau} \int_0^\tau \mathbb{E} \int_{-\infty}^{\infty} \|\mathcal{D}_r Y(\hat{s},\omega)\|^2 dr d\hat{s}, \end{aligned} \tag{2.8}$$

where we use the periodicity $Y(\hat{s},\omega)$ and the norm preserving property Lemma 2.3 about $\mathbb{E}\int_{-\infty}^{+\infty} \|\mathcal{D}_r Y(\hat{s},\omega)\|^2 dr$. Moreover, it is easy to see that (2.8) tends to 0 when $l \to \infty$. Analogously, as u is invertible,

$$\begin{aligned} P^+ Y(t,\omega) &= P^+ u(t - l\tau, t, Y(t, \theta_{l\tau}\omega), \theta_{l\tau}\omega) \\ &= \mathrm{e}^{-Al\tau} P^- Y(t, \theta_{l\tau}\omega) - \int_t^{t+l\tau} \mathrm{e}^{A(t-\hat{s})} P^+ F(\hat{s}, Y(\hat{s},\omega)) d\hat{s} \\ & \quad - \sum_{k=1}^{M} \int_t^{t+l\tau} \mathrm{e}^{A(t-\hat{s})} P^+ B_k(\hat{s}, Y(\hat{s},\omega)) \circ dW_{\hat{s}}^k \\ &\to -\int_t^{\infty} \mathrm{e}^{A(t-\hat{s})} P^+ F(\hat{s}, Y(\hat{s},\omega)) d\hat{s} - \sum_{k=1}^{M} \int_t^{\infty} \mathrm{e}^{A(t-\hat{s})} P^- B_k(\hat{s}, Y(\hat{s},\omega)) \circ dW_{\hat{s}}^k \end{aligned}$$

in L^2-norm as $l \to +\infty$. $\square$

In the general multiplicative noise case, it remains open to solve Eqn. (1.6). We will solve the linear noise case in the next two sections.

3 Linear noise: the exponential dichotomy and IHRIEs

Consider the following τ-periodic semilinear SDE of Stratonovich type with multiplicative linear noise,

$$\begin{cases} du(t) = Au(t)\,dt + F(t, u(t))\,dt + \sum\limits_{k=1}^{M} B_k u(t) \circ dW_t^k, & t \geq s \\ u(s) = x \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

where $A, \{B_k,\ 1 \leq k \leq M\}$ are in $\mathcal{L}(\mathbb{R}^d)$, $W_t := (W_t^1, W_t^2, \cdots, W_t^M)$, $t \in \mathbb{R}$, is an M-dimensional Brownian motion under the filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}^t)_{t\in\mathbb{R}}, \mathbb{P})$. In addition, we assume that

Condition (C). The matrices A, A^*, B_k, and B_k^* are mutually commutative.

Now define a random evolution operator $\Phi : \mathbb{R}^+ \times \Omega \to \mathcal{L}(\mathbb{R}^d)$ by

$$\begin{cases} d\Phi(t) = A\Phi(t)\,dt + \sum\limits_{k=1}^{M} B_k \Phi(t) \circ dW_t^k, & t \geq 0 \\ \Phi(0) = I \in \mathcal{L}(\mathbb{R}^d), \end{cases} \tag{3.2}$$

which is a cocycle over the metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ ([2],[6],[12],[20],[22]). Due to the commutative property of A and B_k, Φ can be written in the explicit form as

$$\Phi(t, \omega) = \exp\left\{At + \textstyle\sum_{k=1}^{M} B_k W_t^k\right\}.$$

Recall that the solution of (3.1) via (3.2) can be written as (c.f. [23])

$$u(t, s, x, \omega) = \Phi(t - s, \theta_s \omega)x + \int_s^t \Phi(t - \hat{s}, \theta_{\hat{s}}\omega) F(\hat{s}, u(\hat{s}, s, x, \omega))\,d\hat{s}, \quad t \geq s. \tag{3.3}$$

Remark 3.1. *Though being defined only on $\mathbb{R}^+$ in the above, Φ can be extended to $\mathbb{R}^-$ in the finite dimentional case via the relation $\Phi(t, \omega) = \Phi(-t, \theta_t \omega)^{-1}$ when $t \leq 0$ as it is measurably invertible. Here $\Phi(t, \omega)^{-1}$ is uniquely defined and satisfies ([20])*

$$d\Phi(t)^{-1} = -A\Phi(t)^{-1}\,dt + \sum_{k=1}^{M} B_k^2 \Phi(t)^{-1}\,dt - \sum_{k=1}^{M} B_k \Phi(t)^{-1} \circ dW_t^k, \quad t \geq 0.$$

Note it is not hard to check that Φ is a perfect two-sided linear cocycle, so it satisfies the multiplicative ergodic theorem (MET) in Euclidean space ([2]). The proof is postponed to the Appendix.

Lemma 3.2. (Exponential dichotomy) Suppose that $\frac{A+A^*}{2}$ has only nonzero eigenvalues with the order $\mu_p < \mu_{p-1} < \cdots < \mu_{m+1} < 0 < \mu_m < \cdots < \mu_1$, $p \leq d$, and the corresponding eigenspaces $E_p, \cdots, E_1$ with multiplicity $d_i := \dim E_i$. Here $\sum_{i=1}^{p} d_i = d$. Then

(i) There exists a non-random splitting

$$\mathbb{R}^d = E_p \oplus E_{p-1} \oplus \cdots \oplus E_{m+1} \cdots \oplus E_1 \quad \mathbb{P} - \text{a.s.},$$

and
$$\mu_i = \lim_{t\to\pm\infty} \frac{1}{t}\log|\Phi(t,\omega)x|, \text{ for } x \in E_i \setminus \{0\},$$
is the Lyapunov exponent of Φ, with the corresponding multiplicity d_i. Moreover, $\mathbb{R}^d$ can be decomposed as
$$\mathbb{R}^d = E^- \oplus E^+,$$
where $E^- = E_p \oplus E_{p-1} \oplus \cdots \oplus E_{m+1}$ is generated by the eigenvectors with negative eigenvalues, while $E^+ = E_m \oplus E_{m-1} \oplus \cdots \oplus E_1$ is generated by the eigenvectors with positive eigenvalues.

(ii) Let $P^\pm : \mathbb{R}^d \to E^\pm$ be the projection onto $E^\pm$ along $E^\mp$. Then
$$\Phi(t,\theta_s\omega)P^\pm = P^\pm\Phi(t,\theta_s\omega) \quad \mathbb{P}-\text{a.s.},$$
with exponential dichotomy on an invariant set $\hat{\Omega}$ of full measure,
$$\begin{cases} \|\Phi(t,\theta_s\omega)P^+\| \le C(\theta_s\omega)e^{\frac{1}{2}\mu_m t} \le C_\Lambda(\omega)e^{\frac{1}{2}\mu_m t}e^{\Lambda|s|}, & t\le 0,\\ \|\Phi(t,\theta_s\omega)P^-\| \le C(\theta_s\omega)e^{\frac{1}{2}\mu_{m+1} t} \le C_\Lambda(\omega)e^{\frac{1}{2}\mu_{m+1} t}e^{\Lambda|s|}, & t\ge 0, \end{cases} \tag{3.4}$$
for any $s\in\mathbb{R}$, where $\|\cdot\|$ denotes the norm on $\mathcal{L}(\mathbb{R}^d)$, and $C(\omega)$ is a tempered random variable from above, Λ is an arbitrary positive number and $C_\Lambda(\omega)$ a positive random variable depending on Λ.

Some elementary but useful results can be derived from (3.4). Their proof is postponed to an Appendix.

Corollary 3.3. *For any $t \ge 0$, and $\hat{s}\in\mathbb{R}$, we have*
$$\mathbb{E}\|P^- - \Phi(t,\theta_{\hat{s}}\cdot)P^-\|^2 = \mathbb{E}\|P^- - \Phi(t,\cdot)P^-\|^2 \le C(|t|+1)e^{2\|A\||t|+2M\|B\|^2|t|}|t|,$$
where C is a generic constant that may depend on M, A, B_k, μ_{m+1}, μ_m, F, and τ, and for any $t\le 0$, and $\hat{s}\in\mathbb{R}$, we have
$$\mathbb{E}\|P^+ - \Phi(t,\theta_{\hat{s}}\cdot)P^+\|^2 = \mathbb{E}\|P^+ - \Phi(t,\cdot)P^+\|^2 \le C(|t|+1)e^{2\|A\||t|+2M\|B\||t|}|t|.$$
Moreover, we have
$$\mathbb{E}\|\Phi(t,\theta_{\hat{s}}\cdot)P^\pm\|^2 = \mathbb{E}\|\Phi(t,\cdot)P^\pm\|^2 \le Ce^{2\|A\||t|+2M\|B\|^2|t|}.$$

We will look for a $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}$-measurable map $Y:\mathbb{R}\times\Omega\to\mathbb{R}^d$ which satisfies the following coupled forward-backward IHRIE,
$$\begin{aligned} Y(t,\omega) &= \int_{-\infty}^t \Phi(t-\hat{s},\theta_{\hat{s}}\omega)P^-F(\hat{s},Y(\hat{s},\omega))d\hat{s}\\ &\quad -\int_t^{+\infty} \Phi(t-\hat{s},\theta_{\hat{s}}\omega)P^+F(\hat{s},Y(\hat{s},\omega))d\hat{s}, \end{aligned} \tag{3.5}$$

for all $\omega \in \Omega$, $t \in \mathbb{R}$. For any $N \in \mathbb{N}$, set the truncation of $\Phi(t, \theta_{\hat{s}}\omega)P^{\pm}$ by N:

$$\Phi^N(t, \theta_{\hat{s}}\omega)P^- := \Phi(t, \theta_{\hat{s}}\omega)P^- \min\left\{1, \frac{N e^{\frac{1}{2}\mu_{m+1}t} e^{\Lambda|\hat{s}|}}{\|\Phi(t, \theta_{\hat{s}}\omega)P^-\|}\right\}, \text{ when } t \geq 0, \tag{3.6}$$

$$\Phi^N(t, \theta_{\hat{s}}\omega)P^+ := \Phi(t, \theta_{\hat{s}}\omega)P^+ \min\left\{1, \frac{N e^{\frac{1}{2}\mu_m t} e^{\Lambda|\hat{s}|}}{\|\Phi(t, \theta_{\hat{s}}\omega)P^+\|}\right\}, \text{when } t \leq 0. \tag{3.7}$$

We first consider a sequence of $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable maps $\{Y^N\}_{N \geq 1}$ defined by solutions of

$$\begin{aligned} Y^N(t, \omega) &= \int_{-\infty}^{t} \Phi^N(t - \hat{s}, \theta_{\hat{s}}\omega)P^- F(\hat{s}, Y^N(\hat{s}, \omega))d\hat{s} \\ &\quad - \int_{t}^{+\infty} \Phi^N(t - \hat{s}, \theta_{\hat{s}}\omega)P^+ F(\hat{s}, Y^N(\hat{s}, \omega))d\hat{s}, \end{aligned} \tag{3.8}$$

for all $\omega \in \Omega$, $t \in \mathbb{R}$. We will develop tools to solve Eqn. (3.5) via Eqn. (3.8). Denote $\mu := \min\{-\mu_{m+1}, \mu_m\}$. Set

$$\Omega_N := \left\{\omega : \sup_{s \in \mathbb{R}} \max\left\{\sup_{t \geq 0} \|\Phi(t, \theta_s\omega)P^-\| e^{-\frac{1}{2}\mu|t| - \Lambda|s|}, \sup_{t \leq 0} \|\Phi(t, \theta_s\omega)P^+\| e^{-\frac{1}{2}\mu|t| - \Lambda|s|}\right\} \leq N\right\}. \tag{3.9}$$

Note that for $\omega \in \Omega_N$, $\Phi = \Phi^N$, and consequently Eqn. (3.5) coincides with Eqn. (3.8). Moreover Lemma 3.2 suggests that $\Omega_N \to \Omega$ as $N \to \infty$. Therefore Y^N is a local solution of Eqn. (3.5).

Note that Stratonovich integral is defined in the sense of convergence in probability. In the multiplicative linear noise case, with Theorem 2.4 in hand, we are able to identify the random periodic solution of (3.1) with the solution of IHRIE (3.5) without assuming Y being Malliavin differentiable.

Theorem 3.4. *Let $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be of class C^3, globally bounded and the Jacobians $\nabla F(t, \cdot)$ globally bounded. Assume $F(t, u) = F(t + \tau, u)$ for some fixed $\tau > 0$. Then a tempered Y such that $Y(t + \tau, \omega) = Y(t, \theta_\tau\omega)$ for any $t \in \mathbb{R}$ $\mathbb{P}$-a.s. is a random periodic solution if and only if Y satisfies (3.5).*

Proof. If Eqn. (3.5) has a solution $Y(\cdot, \omega)$, then from Eqn. (3.5) by using the cocycle property of Φ we have for any $t \geq s$,

$$Y(t, \omega) = \Phi(t - s, \theta_s\omega)Y(s, \omega) + \int_s^t \Phi(t - \hat{s}, \theta_{\hat{s}}\omega)F(\hat{s}, Y(\hat{s}, \omega))d\hat{s}.$$

This is to say that $Y(t, \omega)$ satisfies (3.3) with initial value $Y(s, \omega)$. Now suppose that $u(t, s, \varphi_1, \omega)$ and $u(t, s, \varphi_2, \omega)$ are solutions of Eqn. (3.3) with $\mathcal{F}$-measurable initial values φ_1 and φ_2 respectively. Then

$$\begin{aligned} &|u(t, s, \varphi_1, \omega) - u(t, s, \varphi_2, \omega)|^2 \\ \leq\ & 2\|\Phi(t - s, \theta_s\omega)\|^2 |\varphi_1 - \varphi_2|^2 \end{aligned}$$

$$+2(T-s)\int_s^t \|\Phi(t-\hat{s},\theta_{\hat{s}}\omega)\|^2\|\nabla F\|_\infty^2|u(\hat{s},s,\varphi_1,\omega)-u(\hat{s},s,\varphi_2,\omega)|^2\,d\hat{s},$$

where

$$\|\nabla F\|_\infty^2 := \sup_{t\in\mathbb{R},u\in\mathbb{R}^d}\|\nabla F(t,u)\|^2_{\mathcal{L}(\mathbb{R}^d)}.$$

For any $t>s$,

$$\begin{aligned}\|\Phi(t-s,\theta_s\omega)\| &= \left\|\exp\left\{\frac{1}{2}(A+A^*)(t-s)\right\}\right\|\left\|\exp\left\{\frac{1}{2}\textstyle\sum_{k=1}^M(B_k+B_k^*)(W_t^k-W_s^k)\right\}\right\|\\ &\le \mathrm{e}^{\mu_1(t-s)}\prod_{k=1}^M\exp\left\{\|B\|(2C^k_{\delta,\omega}+|t|^\delta+|s|^\delta)\right\}\\ &\le \mathrm{e}^{\mu_1(t-s)}\exp\left\{2M\|B\|\hat{T}+2\|B\|\textstyle\sum_{k=1}^M C^k_{\delta,\omega}\right\},\end{aligned}$$

where $\hat{T}:=\max\{|T|^\delta+|s|^\delta,2|s|^\delta\}$, and the third line holds due to the fact that there exists Ω_1 of full measure and a constant $\frac{1}{2}<\delta<1$ such that

$$|W_t^k-W_s^k|\le 2C^k_{\delta,\omega}+|t|^\delta+|s|^\delta.$$

Then for any $s\le t\le T$,

$$\begin{aligned}&|u(t,s,\varphi_1,\omega)-u(t,s,\varphi_2,\omega)|^2\\ \le\ & 2H_\omega(T-s)|\varphi_1-\varphi_2|^2+2(T-s)\|\nabla F\|^2_\infty H_\omega(T-s)\int_s^t|u(\hat{s},s,\varphi_1,\omega)-u(\hat{s},s,\varphi_2,\omega)|^2\,d\hat{s},\end{aligned}$$

where

$$H_\omega(T-s)=\mathrm{e}^{2\mu_1(t-s)}\exp\left\{4M\|B\|\hat{T}+4\|B\|\textstyle\sum_{k=1}^M C^k_{\delta,\omega}\right\}.$$

Thus applying the Gronwall's inequality gives

$$|u(t,s,\varphi_1,\omega)-u(t,s,\varphi_2,\omega)|^2\le 2H_\omega(T-s)|\varphi_1-\varphi_2|^2\mathrm{e}^{2\|\nabla F\|^2_\infty H_\omega(T-s)(T-s)^2}\quad \mathbb{P}-\text{a.s.}$$

Now assume that $\varphi_1=\varphi_2$. Then it is easy to see that $u(t,s,\varphi_1,\omega)=u(t,s,\varphi_2,\omega)$ for any $\omega\in\Omega_1$ and $t\in[s,T]$. Hence from $\mathbb{P}(\Omega_1)=1$,

$$\mathbb{P}\left\{u(t,s,\varphi_1,\omega)=u(t,s,\varphi_2,\omega)\ for\ any\ t\in\mathbb{Q}\cap[s,T]\right\}=1,$$

where $\mathbb{Q}$ is the set of rational numbers. By the continuity of $t\to|u(t,s,\varphi_1,\omega)-u(t,s,\varphi_2,\omega)|$, it follows that

$$\mathbb{P}\left\{u(t,s,\varphi_1,\omega)=u(t,s,\varphi_2,\omega)\ for\ any\ \ t\in[s,T]\right\}=1.$$

This implies the uniqueness of solution of SDE (3.5) within a finite time interval $[s,T]$. Then by Theorem 2.4 and the uniqueness of the solution of the initial value problem (3.3), which is equivalent to (3.1),

$$u(t,s,x,\omega)\Big|_{x=Y(s,\omega)}=u(t,s,Y(s,\omega),\omega)=Y(t,\omega).$$

The temperedness of Y follows from the estimates (3.4) and the boundedness of F.

Conversely, assume Eqn. (3.1) has a random periodic solution which is tempered from above. First note for any non-negative integer l, we have by Theorem 2.4,

$$\begin{aligned} Y(t,\omega) &= u(t\pm l\tau, t, Y(t,\theta_{\mp l\tau}\omega), \theta_{\mp l\tau}\omega) \\ &= \Phi(\pm l\tau, \theta_{t\mp l\tau}\omega)Y(t,\theta_{\mp l\tau}\omega) \\ &\quad + \int_t^{t\pm l\tau} \Phi(t\pm l\tau-\hat{s}, \theta_{\hat{s}\mp l\tau}\omega)F(\hat{s}, u(\hat{s}, t, Y(t,\theta_{\mp l\tau}\omega), \theta_{\mp l\tau}\omega))d\hat{s}. \end{aligned}$$

In particular,

$$\begin{aligned} P^- Y(t,\omega) &= P^- u(t+l\tau, t, Y(t,\theta_{-l\tau}\omega), \theta_{-l\tau}\omega) \\ &= \Phi(l\tau, \theta_{t-l\tau}\omega)P^- Y(t,\theta_{-l\tau}\omega) + \int_{t-l\tau}^t \Phi(t-\hat{s}, \theta_{\hat{s}}\omega)P^- F(\hat{s}, Y(\hat{s},\omega))d\hat{s} \\ &\to \int_{-\infty}^t \Phi(t-\hat{s}, \theta_{\hat{s}}\omega)P^- F(\hat{s}, Y(\hat{s},\omega))d\hat{s} \quad \mathbb{P}-\text{a.s.}, \end{aligned}$$

as $l\to+\infty$. The convergence deserves some justifications. The convergence of the first term to 0 as $l\to+\infty$ can be easily drawn from the estimate (3.4) together with the tempered property of $|Y(t,\omega)|$ and $C(\omega)$. The convergence of the second term to the desired integral can be seen from the estimate of Φ and the boundedness of F.

Analogously, as u is invertible,

$$\begin{aligned} P^+ Y(t,\omega) &= P^+ u(t-l\tau, t, Y(t,\theta_{l\tau}\omega), \theta_{l\tau}\omega) \\ &= \Phi(-l\tau, \theta_{t+l\tau}\omega)P^+ Y(t,\theta_{l\tau}\omega) - \int_t^{t+l\tau} \Phi(t-\hat{s}, \theta_{\hat{s}}\omega)P^+ F(\hat{s}, Y(\hat{s},\omega))d\hat{s} \\ &\to -\int_t^{+\infty} \Phi(t-\hat{s}, \theta_{\hat{s}}\omega)P^+ F(\hat{s}, Y(\hat{s},\omega))d\hat{s} \quad \mathbb{P}-\text{a.s.} \end{aligned}$$

as $l\to+\infty$. Therefore we have proved the converse part as $Y=P^+Y+P^-Y$. □

4 The existence of random periodic solutions and periodic measures

After showing the equivalence of random periodic solutions of (3.1) and the solutions of (3.5), it remains to prove the existence of solutions to (3.5). To check the relatively compactness is key to the proof of the the main result. In the following, we present the improved version of the Wiener-Sobolev compact embedding in [13] with less conditions. We provide a brief proof in the Appendix for completeness. This kind of compactness in $L^2(\Omega)$ as a purely random variable version without including time and space variables was investigated in [9],[26], and in $L^2([a,b],L^2(\Omega))$ was obtained in [3].

Theorem 4.1. (Relative Compactness in $C([a,b],L^2(\Omega))$. Consider a sequence $(v_n)_{n\in\mathbb{N}}$ of $C([a,b],L^2(\Omega))$. Suppose that:

(i) $v_n(t,\cdot)\in\mathcal{D}^{1,2}$ and $\sup_{n\in\mathbb{N}}\sup_{t\in[a,b]}||v_n(t,\cdot)||_{1,2}^2<\infty$.

(ii) There exists a constant $C>0$ such that for any $t,s\in[a,b]$,

$$\sup_n \mathbb{E}|v_n(t)-v_n(s)|^2<C|t-s|.$$

(iii) (3i) There exists a constant $C>0$ such that for any $h_1\in\mathbb{R}$, and any $t\in[a,b]$,

$$\sup_n \int_{\mathbb{R}} \mathbb{E}|\mathcal{D}_{r+h_1}v_n(t)-\mathcal{D}_r v_n(t)|^2 dr<C|h_1|.$$

(3ii) For any $\epsilon>0$, there exists $-\infty<\alpha<\beta<+\infty$ such that

$$\sup_n \sup_{t\in[a,b]} \int_{\mathbb{R}\setminus[\alpha,\beta]} \mathbb{E}|\mathcal{D}_r v_n(t)|^2 dr<\epsilon.$$

Then $\{v_n,\ n\in\mathbb{N}\}$ is relatively compact in $C([a,b],L^2(\Omega))$.

The local existence theorem of random periodic solutions is presented below.

Proposition 4.2. *Let $F:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ be in $C^3(\mathbb{R}^{d+1})$, globally bounded and the Jacobian $\nabla F(t,\cdot)$ be globally bounded, and $F(t,u)=F(t+\tau,u)$ for some fixed $\tau>0$, and Condition (C) holds. Then there exists at least one $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}$-measurable map $Y^N:\mathbb{R}\times\Omega\to\mathbb{R}^d$ satisfying Eqn. (3.8) and $Y^N(t+\tau,\omega)=Y^N(t,\theta_\tau\omega)$ for any $t\in\mathbb{R}$ and $\omega\in\Omega$.*

Remark 4.3. *It will be clear from the proof of this theorem that the commutativity Condition (C) is necessary only in the case when A is hyperbolic with at least one eigenvalue having a positive real part and at least one eigenvalue having negative real part, as otherwise, projection operators are not needed.*

The idea of its proof is to find a fixed point in some specific Banach space under Schauder's fixed point argument [13]. The proof of this theorem is quite long, so we break into many parts. Firstly we define a Banach space $C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$

$$C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d)):=\{f\in C^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d)):\ \text{for any } t\in\mathbb{R}, f(t+\tau,\omega)=f(t,\theta_\tau\omega)\}. \quad (4.1)$$

Here the norm of the metric space $C^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$ is given as follows,

$$\|f\|_\Lambda:=\sup_{t\in\mathbb{R}} \mathrm{e}^{-2\Lambda|t|}\|f(t,\cdot)\|_{L^2(\Omega,\mathbb{R}^d)},$$

which is indeed a weighted norm with $0<\Lambda<\frac{1}{4}\mu=\frac{1}{4}\min\{-\mu_{m+1},\mu_m\}$. Define a map $\mathcal{M}^N$: for any $Y^N\in C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$,

$$\begin{aligned}\mathcal{M}^N(Y^N)(t,\omega) &= \int_{-\infty}^t \Phi^N(t-\hat{s},\theta_{\hat{s}}\omega)P^-F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\\ &\quad -\int_t^{+\infty}\Phi^N(t-\hat{s},\theta_{\hat{s}}\omega)P^+F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}. \quad (4.2)\end{aligned}$$

Lemma 4.4. *Under the conditions of Proposition 4.2, the map*

$$\mathcal{M}^N : C_\tau^\Lambda(\mathbb{R}, L^2(\Omega, \mathbb{R}^d)) \to C_\tau^\Lambda(\mathbb{R}, L^2(\Omega, \mathbb{R}^d))$$

is continuous.

Proof. ***Step 1***: We now show that $\mathcal{M}^N$ maps $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega, \mathbb{R}^d))$ into itself.

(A) We first verify that for any $Y^N \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega, \mathbb{R}^d))$,

$$\sup_{t\in\mathbb{R}} \mathrm{e}^{-2\Lambda|t|}\mathbb{E}|\mathcal{M}^N(Y^N)(t,\cdot)|^2 < \infty.$$

Actually by (3.6) and (3.7) we have that

$$\begin{aligned}
&\mathrm{e}^{-2\Lambda|t|}\mathbb{E}|\mathcal{M}^N(Y^N)(t,\cdot)|^2\\
\le\ & 2\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\Big|\int_{-\infty}^t \Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y^N)d\hat{s}\Big|^2 + 2\mathrm{e}^{-\Lambda|t|}\mathbb{E}\Big|\int_t^{+\infty} \Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y^N)d\hat{s}\Big|^2\\
\le\ & 2\mathrm{e}^{-2\Lambda|t|}\|F\|_\infty^2\Big\{\mathbb{E}\Big(\int_{-\infty}^t \|\Phi^N_{t-\hat{s},\hat{s}}P^-\|d\hat{s}\Big)^2 + \mathbb{E}\Big(\int_t^{+\infty} \|\Phi^N_{t-\hat{s},\hat{s}}P^+\|d\hat{s}\Big)^2\Big\}\\
\le\ & 2N^2\|F\|_\infty^2\mathrm{e}^{-2\Lambda|t|}\Big\{\Big(\int_{-\infty}^t \mathrm{e}^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathrm{e}^{\Lambda|\hat{s}|}d\hat{s}\Big)^2 + \Big(\int_t^{+\infty} \mathrm{e}^{\frac{1}{2}\mu_m(t-\hat{s})}\mathrm{e}^{\Lambda|\hat{s}|}d\hat{s}\Big)^2\Big\}.
\end{aligned}$$

Here $\Phi_{t,\hat{s}}P^\pm$ is the shorthand for $\Phi(t,\theta_{\hat{s}}\omega)P^\pm$, and $\Phi_tP^\pm$ is the shorthand for $\Phi(t,\omega)P^\pm$. Note that $\mathrm{e}^{\Lambda|\hat{s}|} \le \mathrm{e}^{-\Lambda\hat{s}} + \mathrm{e}^{\Lambda\hat{s}}$, $\mathrm{e}^{-2\Lambda|t|} \le \mathrm{e}^{-2\Lambda t}$ and $\mathrm{e}^{-2\Lambda|t|} \le \mathrm{e}^{2\Lambda t}$ for all $\hat{s}, t \in \mathbb{R}$. The first integral in the above can be estimated as

$$\begin{aligned}
&\mathrm{e}^{-2\Lambda|t|}\Big(\int_{-\infty}^t \mathrm{e}^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathrm{e}^{\Lambda|\hat{s}|}d\hat{s}\Big)^2\\
\le\ & \mathrm{e}^{-2\Lambda|t|}\Big(\int_{-\infty}^t \mathrm{e}^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathrm{e}^{-\Lambda\hat{s}}d\hat{s}\Big)^2 + \Big(\int_{-\infty}^t \mathrm{e}^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathrm{e}^{\Lambda\hat{s}}d\hat{s}\Big)^2\\
\le\ & \Big(\int_{-\infty}^t \mathrm{e}^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}d\hat{s}\Big)^2 + \Big(\int_{-\infty}^t \mathrm{e}^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}d\hat{s}\Big)^2\\
\le\ & \frac{1}{(\mu_{m+1}+2\Lambda)^2} + \frac{1}{(\mu_{m+1}-2\Lambda)^2}.
\end{aligned}$$

The second integral integral can be estimated similarly. Putting them together, we have

$$\begin{aligned}
&\mathrm{e}^{-2\Lambda|t|}\mathbb{E}|\mathcal{M}^N(Y^N)(t,\cdot)|^2\\
\le\ & 8N^2\|F\|_\infty^2\Big\{\frac{1}{(\mu_{m+1}+2\Lambda)^2} + \frac{1}{(\mu_{m+1}-2\Lambda)^2} + \frac{1}{(\mu_m+2\Lambda)^2} + \frac{1}{(\mu_m-2\Lambda)^2}\Big\}.
\end{aligned}$$

(B) Next we show that $\mathcal{M}^N(Y^N)(\cdot,\omega)$ is continuous from $\mathbb{R}$ to $L^2(\Omega,\mathbb{R}^d)$ for any given $Y^N \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega,\mathbb{R}^d))$. First note for any t_1, $t_2 \in \mathbb{R}$ with $t_1 \le t_2$,

$$\mathbb{E}|\mathcal{M}^N(Y^N)(t_1) - \mathcal{M}^N(Y^N)(t_2)|^2$$

$$
\begin{aligned}
&\leq \quad 4\mathbb{E}\Big[\Big|\int_{-\infty}^{t_1}(\Phi^N_{t_1-\hat{s},\hat{s}}P^- - \Phi^N_{t_2-\hat{s},\hat{s}}P^-)F(\hat{s},Y^N)d\hat{s}\Big|^2 + \Big|\int_{t_1}^{t_2}\Phi^N_{t_2-\hat{s},\hat{s}}P^-F(\hat{s},Y^N)d\hat{s}\Big|^2 \\
&\qquad + \Big|\int_{t_2}^{+\infty}(\Phi^N_{t_1-\hat{s},\hat{s}}P^+ - \Phi^N_{t_2-\hat{s},\hat{s}}P^+)F(\hat{s},Y^N)d\hat{s}\Big|^2 + \Big|\int_{t_1}^{t_2}\Phi^N_{t_1-\hat{s},\hat{s}}P^+F(\hat{s},Y^N)d\hat{s}\Big|^2\Big] \\
&=: \quad \sum_{i=1}^4 T_i.
\end{aligned}
$$

It is easy to check that

$$
\begin{aligned}
T_2 = 4\mathbb{E}\Big|\int_{t_1}^{t_2}\Phi^N_{t_2-\hat{s},\hat{s}}P^-F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 &\leq 4N^2\|F\|^2_\infty\Big(\int_{t_1}^{t_2}e^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2 \\
&\leq 4N^2\|F\|^2_\infty \max\{e^{2\Lambda|t_2|}, e^{2\Lambda|t_1|}\}|t_2-t_1|^2,
\end{aligned}
$$

and similarly

$$
T_4 = 4\mathbb{E}\Big|\int_{t_1}^{t_2}\Phi^N_{t_1-\hat{s},\hat{s}}P^+F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 \leq 4N^2\|F\|^2_\infty \max\{e^{2\Lambda|t_2|}, e^{2\Lambda|t_1|}\}|t_2-t_1|^2.
$$

As for T_1, we have the following inequalities through the estimates in Lemma 3.3,

$$
\begin{aligned}
T_1 \quad &:= \quad 4\mathbb{E}\Big|\int_{-\infty}^{t_1}(\Phi^N_{t_1-\hat{s},\hat{s}}P^- - \Phi^N_{t_2-\hat{s},\hat{s}}P^-)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\Big|^2 \\
&\leq \quad 8\mathbb{E}\Big|\int_{-\infty}^{t_1}(\Phi_{t_1-\hat{s},\hat{s}}P^- - \Phi_{t_2-\hat{s},\hat{s}}P^-)\min\Big\{1,\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\Big\}F(\hat{s},Y^N)d\hat{s}\Big|^2 \\
&+ \quad 8\mathbb{E}\Big|\int_{-\infty}^{t_1}\Phi_{t_2-\hat{s},\hat{s}}P^-\Bigg(\min\Big\{1,\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\Big\} \\
&\qquad\qquad - \min\Big\{1,\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}\Big\}\Bigg)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\Big|^2.
\end{aligned}
$$

By using inequality $|\min\{1,a\}-\min\{1,b\}| \leq |a-b|$ whenever $a,b \geq 0$, so for $s < t_1 < t_2$ we have

$$
\begin{aligned}
&\left|\min\Big\{1,\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\Big\} - \min\Big\{1,\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}\Big\}\right| \\
\leq\ &\left|\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|} - \frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}\right| \\
\leq\ &\left|\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|} - \frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\right| + \left|\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|} - \frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}\right| \\
\leq\ &\frac{Ne^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\Big(e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})} - e^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}\Big) + Ne^{\Lambda|\hat{s}|}e^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}\left|\frac{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\| - \|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}{\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\right| \\
\leq\ &\frac{Ne^{\Lambda|\hat{s}|}e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}}{\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}\Big((1-e^{\frac{1}{2}\mu_{m+1}(t_2-t_1)})\|\Phi_{t_2-t_1,t_1}P^-\| + \|\Phi_{t_2-t_1,t_1}P^- - P^-\|\Big).
\end{aligned}
$$

Therefore

$$\begin{aligned}
T_1 &\leq 384N^2\|F\|_\infty^2 \mathrm{e}^{2\Lambda|t_1|}\Big(\frac{1}{|\mu_{m+1}+2\Lambda|^2}+\frac{1}{|\mu_{m+1}-2\Lambda|^2}\Big)\\
&\quad\cdot[\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^- - P^-\|^2+\mu_{m+1}^2(t_2-t_1)^2\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^-\|^2]\\
&\leq CN^2\|F\|_\infty^2\mathrm{e}^{2\Lambda|t_1|}\mathrm{e}^{2\|A\||t_2-t_1|+2M\|B\|^2|t_2-t_1|}\Big(\frac{1}{|\mu_{m+1}+2\Lambda|^2}+\frac{1}{|\mu_{m+1}-2\Lambda|^2}\Big)\\
&\quad\cdot[(1+\mu_{m+1}^2)|t_2-t_1|^2+|t_2-t_1|],
\end{aligned}$$

where the last inequality follows from Lemma 3.3. Similarly,

$$\begin{aligned}
T_3 &:= 4\mathbb{E}\Big|\int_{t_2}^{+\infty}(\Phi^N_{t_1-\hat{s},\hat{s}}P^+ - \Phi^N_{t_2-\hat{s},\hat{s}}P^+)F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2\\
&\leq CN^2\|F\|_\infty^2\mathrm{e}^{2\Lambda|t_2|}\mathrm{e}^{2\|A\||t_2-t_1|+2M\|B\|^2|t_2-t_1|}\Big(\frac{1}{|\mu_m+2\Lambda|^2}+\frac{1}{|\mu_m-2\Lambda|^2}\Big)\\
&\quad\cdot[(1+\mu_m^2)|t_2-t_1|^2+|t_2-t_1|].
\end{aligned}$$

(C) We show that $\mathcal{M}^N(Y^N)(t,\theta_{\pm\tau}\omega)=\mathcal{M}^N(Y^N)(t\pm\tau,\omega)$: similar as in [13], as $Y^N(t+\tau,\omega)=Y^N(t,\theta_\tau\omega)$, so

$$\begin{aligned}
&\mathcal{M}^N(Y^N)(t,\theta_\tau\omega)\\
=&\int_{-\infty}^t\Phi^N_{t-\hat{s},\hat{s}+\tau}P^-F(\hat{s},Y^N(\hat{s},\theta_\tau\omega))d\hat{s}-\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}+\tau}P^+F(\hat{s},Y^N(\hat{s},\theta_\tau\omega))d\hat{s}\\
=&\int_{-\infty}^t\Phi^N_{(t+\tau)-(\hat{s}+\tau),\hat{s}+\tau}P^-F(\hat{s}+\tau,Y^N(\hat{s}+\tau,\omega))d\hat{s}\\
&-\int_t^{+\infty}\Phi^N_{(t+\tau)-(\hat{s}+\tau),\hat{s}+\tau}P^+F(\hat{s}+\tau,Y^N(\hat{s}+\tau,\omega))d\hat{s}\\
=&\int_{-\infty}^{t+\tau}\Phi^N_{(t+\tau)-\hat{h},\hat{h}}P^-F(\hat{h},Y^N(\hat{h},\omega))d\hat{h}-\int_{t+\tau}^{+\infty}\Phi^N_{(t+\tau)-\hat{h},\hat{h}}P^+F(\hat{h},Y^N(\hat{h},\omega))d\hat{h}\\
=&\mathcal{M}^N(Y^N)(t+\tau,\omega).
\end{aligned}$$

Thus we completed the Step 1 and proved that $\mathcal{M}^N$ maps $C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$ into itself.

Step 2: We now check the continuity of the map $\mathcal{M}^N: C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))\to C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$. For $Y_1^N,Y_2^N\in C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$ and $t\in[j\tau,(j+1)\tau)$ for some $j\in\mathbb{Z}$, we have

$$\begin{aligned}
&\mathrm{e}^{-2\Lambda|t|}\mathbb{E}|\mathcal{M}^N(Y_1^N)(t,\cdot)-\mathcal{M}^N(Y_2^N)(t,\cdot)|^2\\
\leq&\ 2\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\Big|\int_{-\infty}^t\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y_1^N(\hat{s},\cdot))d\hat{s}-\int_{-\infty}^t\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y_2^N(\hat{s},\cdot))d\hat{s}\Big|^2\\
&+2\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\Big|\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y_1^N(\hat{s},\cdot))d\hat{s}-\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y_2^N(\hat{s},\cdot))d\hat{s}\Big|^2,\\
:=&\ \hat{T}_1+\hat{T}_2.
\end{aligned}$$

By using the Cauchy-Schwarz inequality we have that

$$\hat{T}_1 \leq 2\|\nabla F\|_\infty^2\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\Big(\int_{-\infty}^t\|\Phi^N_{t-\hat{s},\hat{s}}P^-\||Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|d\hat{s}\Big)^2$$

$$
\begin{aligned}
&\leq 4N^2\|\nabla F\|_\infty^2\mathbb{E}\Big(\int_{-\infty}^{t} e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|d\hat{s}\Big)^2\\
&\quad +4N^2\|\nabla F\|_\infty^2\mathbb{E}\Big(\int_{-\infty}^{t} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|d\hat{s}\Big)^2\\
&\leq \frac{8}{|\mu_{m+1}-2\Lambda|}N^2\|\nabla F\|_\infty^2\int_{-\infty}^{t} e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\\
&\quad +\frac{8}{|\mu_{m+1}+2\Lambda|}N^2\|\nabla F\|_\infty^2\int_{-\infty}^{t} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}.
\end{aligned}
$$

Note that $\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2$ is a nonegative periodic function in $C^\Lambda(\mathbb{R})$ with period τ as

$$
\mathbb{E}|Y_1^N(\hat{s}+\tau,\cdot)-Y_2^N(\hat{s}+\tau,\cdot)|^2=\mathbb{E}|Y_1^N(\hat{s},\theta_\tau\cdot)-Y_2^N(\hat{s},\theta_\tau\cdot)|^2=\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2.
$$

Then we have

$$
\begin{aligned}
&\int_{-\infty}^{t} e^{(\frac{1}{2}\mu_{m+1}\pm\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\\
&\leq \sup_{s\in[0,\tau)}\mathbb{E}|Y_1^N(s,\cdot)-Y_2^N(s,\cdot)|^2\int_{-\infty}^{t} e^{(\frac{1}{2}\mu_{m+1}\pm\Lambda)(t-\hat{s})}d\hat{s}\\
&\leq \frac{2e^{2\Lambda\tau}}{|\mu_{m+1}\pm2\Lambda|}\sup_{s\in[0,\tau)} e^{-2\Lambda|s|}\mathbb{E}|Y_1^N(s,\cdot)-Y_2^N(s,\cdot)|^2.
\end{aligned}
$$

This leads to

$$
\hat{T}_1 \leq 6N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}\Big\{\frac{1}{(\mu_{m+1}-2\Lambda)^2}+\frac{1}{(\mu_{m+1}+2\Lambda)^2}\Big\}\sup_{\hat{s}\in\mathbb{R}} e^{-2\Lambda|\hat{s}|}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2,
$$

Similarly

$$
\hat{T}_2 \leq 16N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}\Big\{\frac{1}{(\mu_{m}-2\Lambda)^2}+\frac{1}{(\mu_{m}+2\Lambda)^2}\Big\}\sup_{\hat{s}\in\mathbb{R}} e^{-2\Lambda|\hat{s}|}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2.
$$

Therefore the continuity of $\mathcal{M}^N: C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))\to C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$ is verified. □

Remark 4.5. *One can see that it is crucial to use the truncation of the tempered random variable $C(\omega)$ in the Step 2 of the proof. Otherwise, it would be difficult to separate $\|\Phi^N_{t-\hat{s},\hat{s}}P^\pm\|^2$ and $|Y_1^N(\hat{s},\omega)-Y_2^N(\hat{s},\omega)|^2$ inside the integrals in $\hat{T}_1$ and $\hat{T}_2$, where Hölder's inequality seems losing its power here. Needless to say that a key step to make it work is to remove the truncation eventually.*

Lemma 4.6. *Given $\Phi^N(t,\theta_{\hat{s}}\omega)P^\pm$ defined by (3.7) and (3.6), the Malliavin derivatives of $\Phi^N(t,\theta_{\hat{s}}\omega)P^\pm$ with respect to the l-th Brownian motion, $l\in\{1,2,\cdots,M\}$, are given by: when $t\geq 0$*

$$
\begin{aligned}
&\mathcal{D}_r^l\Phi^N(t,\theta_{\hat{s}}\omega)P^-\\
&= \chi_{\{\hat{s}\leq r\leq t+\hat{s}\}}(r)\Bigg\{B_l\Phi(t,\theta_{\hat{s}}\omega)P^-\min\Bigg\{1,\frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&-\chi_{\{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|>N\mathrm{e}^{\frac{1}{2}\mu_{m+1}t}\mathrm{e}^{\Lambda|\hat{s}|}\}}(\omega)\frac{N\mathrm{e}^{\frac{1}{2}\mu_{m+1}t}\mathrm{e}^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|^3}\\
&\cdot\Big(\sum_{i,j=1}^d(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij}\sum_{k=1}^d(B_l)_{ik}(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{kj}\Big)\Phi(t,\theta_{\hat{s}}\omega)P^-\Bigg\}
\end{aligned}
\tag{4.3}
$$

with the estimate that

$$
\|\mathcal{D}_r^l\Phi^N(t,\theta_{\hat{s}}\omega)P^-\|\le(1+d^3)\|B\|N\mathrm{e}^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathrm{e}^{\Lambda|\hat{s}|},\tag{4.4}
$$

and when $t\le0$

$$
\begin{aligned}
&\mathcal{D}_r^l\Phi^N(t,\theta_{\hat{s}}\omega)P^+\\
=\;&\chi_{\{t+\hat{s}\le r\le\hat{s}\}}(r)\Bigg\{-B_l\Phi(t,\theta_{\hat{s}}\omega)P^+\min\Bigg\{1,\frac{N\mathrm{e}^{\frac{1}{2}\mu_m t}\mathrm{e}^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^+\|}\Bigg\}\\
&+\chi_{\{\|\Phi(t,\theta_{\hat{s}}\omega)P^+\|>N\mathrm{e}^{\frac{1}{2}\mu_m t}\mathrm{e}^{\Lambda|\hat{s}|}\}}(\omega)\frac{N\mathrm{e}^{\frac{1}{2}\mu_m t}\mathrm{e}^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^+\|^3}\\
&\cdot\Big(\sum_{i,j=1}^d(\Phi(t,\theta_{\hat{s}}\omega)P^+)_{ij}\sum_{k=1}^d(B_l)_{ik}(\Phi(t,\theta_{\hat{s}}\omega)P^+)_{kj}\Big)\Phi(t,\theta_{\hat{s}}\omega)P^+\Bigg\}
\end{aligned}
\tag{4.5}
$$

with the estimate that

$$
\|\mathcal{D}_r^l\Phi^N(t,\theta_{\hat{s}}\omega)P^+\|\le(1+d^3)\|B\|N\mathrm{e}^{\frac{1}{2}\mu_m(t-\hat{s})}\mathrm{e}^{\Lambda|\hat{s}|}.\tag{4.6}
$$

Proof. We can calculate the Malliavin derivatives of Φ^N by the chain rule: when $t\ge0$, from Proposition 1.2.3 and Proposition 1.2.4 in [25] (or directly obtained from the proof of Proposition 2.1.10 in [25]), we know that $\varphi(F):=\min\{1,F\}\in\mathcal{D}^{1,2}$ if $F\in\mathcal{D}^{1,2}$, and for fixed t and s we have that

$$
\mathcal{D}_r^l\min\Bigg\{1,\frac{N\mathrm{e}^{\frac{1}{2}\mu_{m+1}t}\mathrm{e}^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\Bigg\}=\chi_{\{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|>N\mathrm{e}^{\frac{1}{2}\mu_{m+1}t}\mathrm{e}^{\Lambda|\hat{s}|}\}}(\omega)\mathcal{D}_r^l\frac{N\mathrm{e}^{\frac{1}{2}\mu_{m+1}t}\mathrm{e}^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|},\tag{4.7}
$$

Thus, for $l\in\{1,2,\cdots,M\}$,

$$
\begin{aligned}
&\mathcal{D}_r^l\Phi^N(t,\theta_{\hat{s}}\omega)P^-\\
=\;&\mathcal{D}_r^l\big(\Phi(t,\theta_{\hat{s}}\omega)P^-\big)\min\Bigg\{1,\frac{N\mathrm{e}^{\frac{1}{2}\mu_{m+1}t}\mathrm{e}^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\Bigg\}+\Phi(t,\theta_{\hat{s}}\omega)P^-\mathcal{D}_r^l\min\Bigg\{1,\frac{N\mathrm{e}^{\frac{1}{2}\mu_{m+1}t}\mathrm{e}^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\Bigg\}\\
=\;&\mathcal{D}_r^l\left(\exp\{At+\textstyle\sum_{k=1}^M B_k\theta_{\hat{s}}(W_t)\}P^-\right)\min\Bigg\{1,\frac{N\mathrm{e}^{\frac{1}{2}\mu_{m+1}t}\mathrm{e}^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\Bigg\}\\
&-\chi_{\{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|>N\mathrm{e}^{\frac{1}{2}\mu_{m+1}t}\mathrm{e}^{\Lambda|\hat{s}|}\}}(\omega)\Phi(t,\theta_{\hat{s}}\omega)P^-\frac{N\mathrm{e}^{\frac{1}{2}\mu_{m+1}t}\mathrm{e}^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|^2}\mathcal{D}_r^l\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|.
\end{aligned}
$$

Note now the equivalence of the matrix norm

$$
\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|:=\sqrt{\sum_{i,j=1}^d(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij}^2},
$$

where $(J)_{ij}$ stands for the ijth element of the matrix J, and

$$\mathcal{D}_r^l(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij} = \sum_{k=1}^d (B_l)_{ik}(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{kj}.$$

Thus by the chain rule we have

$$\begin{aligned}\mathcal{D}_r^l\|\Phi(t,\theta_{\hat{s}}\omega)P^-\| &= \frac{1}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\sum_{i,j=1}^d (\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij}\mathcal{D}_r^l(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij} \\ &= \frac{1}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\Big(\sum_{i,j=1}^d (\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij}\sum_{k=1}^d (B_l)_{ik}(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{kj}\Big).\end{aligned}$$

It is easy to verify (4.4). When $t \le 0$, (4.5) and (4.6) can be derived analogously. □

Next we introduce a subset of $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega,\mathbb{R}^d))$ as follows,

$$\begin{aligned}&C_\tau^\Lambda(\mathbb{R},\mathcal{D}^{1,2}) \\ := \ &\Big\{f\in C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d)) : \ f|_{[0,\tau)}\in C([0,\tau),\mathcal{D}^{1,2}), \sup_{t\in[0,\tau)}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l f(t,\cdot)|^2dr<\infty, \\ &\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{1}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_{r+\delta}^l f(t,\cdot)-\mathcal{D}_r^l f(t,\cdot)|^2dr<\infty, \ l\in\{1,\cdots,M\}\Big\}.\end{aligned}$$

Lemma 4.7. *Under the conditions of Proposition 4.2, we have*

$$\mathcal{M}^N(C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2}))\subset C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2}).$$

Moreover, $\mathcal{M}^N(C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2})|_{[0,\tau)}$ is relatively compact in $C([0,\tau),L^2(\Omega,\mathbb{R}^d))$.

Proof. ***Step 1***: In this step we are going to present that $\mathcal{M}^N$ maps $C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2})$ into itself.

(i) First we have $\mathcal{M}^N(C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d)))\subset C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$: the argument here is the same as in ***Step 1*** in the proof of Lemma 4.4.

(ii) Next to illustrate that for any $t\in[0,\tau)$, $l\in\{1,\cdots,M\}$ and any $Y^N\in C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2})$,

$$\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\cdot)|^2dr<+\infty.$$

By the chain rule, (4.3) and (4.5), the Malliavin derivative of $\mathcal{M}^N(Y^N)(t,\omega)$ is given as:

$$\begin{aligned}\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\omega) &= \int_{-\infty}^r \chi_{\{r\le t\}}(r)\mathcal{D}_r^l(\Phi_{t-\hat{s},\hat{s}}^N P^-)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s} \\ &\quad -\int_r^{+\infty}\chi_{\{r\ge t\}}(r)\mathcal{D}_r^l(\Phi_{t-\hat{s},\hat{s}}^N P^+)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s} \\ &\quad +\int_{-\infty}^t \Phi_{t-\hat{s},\hat{s}}^N P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}_r^l Y^N(\hat{s},\omega)d\hat{s}\end{aligned}$$

$$-\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_rY^N(\hat{s},\omega)d\hat{s}. \qquad (4.8)$$

Then we get for any $t\in\mathbb{R}$ the following L^2-estimation,

$$\begin{aligned}
& \mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2dr \\
= \;& \mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_{-\infty}^t|\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2dr+\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_t^{+\infty}|\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2dr \\
\leq \;& 3\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_{-\infty}^t\Big|\int_{-\infty}^r\mathcal{D}^l_r(\Phi^N_{t-\hat{s},\hat{s}}P^-)F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2dr \\
& +3\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_t^{+\infty}\Big|\int_r^{+\infty}\mathcal{D}^l_r(\Phi^N_{t-\hat{s},\hat{s}}P^+)F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2dr \\
& +3\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big|\int_{-\infty}^t\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))\mathcal{D}^l_rY^N(\hat{s},\cdot)d\hat{s}\Big|^2dr \\
& +3\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big|\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\cdot))\mathcal{D}^l_rY^N(\hat{s},\cdot)d\hat{s}\Big|^2dr \\
= \;& :\sum_{i=1}^4L_i.
\end{aligned}$$

Applying Lemma 4.6, we have that

$$\begin{aligned}
L_1 \leq \;& 6\|B\|^2N^2\|F\|^2_\infty(1+d^3)^2\mathrm{e}^{-2\Lambda|t|}\int_{-\infty}^t\Big(\int_{-\infty}^r\mathrm{e}^{\frac12\mu_{m+1}(t-\hat{s})}\mathrm{e}^{\Lambda|\hat{s}|}d\hat{s}\Big)^2dr \\
\leq \;& 12\|B\|^2N^2\|F\|^2_\infty(1+d^3)^2\int_{-\infty}^t\mathrm{e}^{(\mu_{m+1}+2\Lambda)(t-r)}\Big(\int_{-\infty}^r\mathrm{e}^{(\frac12\mu_{m+1}+\Lambda)(r-\hat{s})}d\hat{s}\Big)^2dr \\
& +12\|B\|^2N^2\|F\|^2_\infty(1+d^3)^2\int_{-\infty}^t\mathrm{e}^{(\mu_{m+1}-2\Lambda)(t-r)}\Big(\int_{-\infty}^r\mathrm{e}^{(\frac12\mu_{m+1}-\Lambda)(r-\hat{s})}d\hat{s}\Big)^2dr \\
\leq \;& 48\|B\|^2N^2\|F\|^2_\infty(1+d^3)^2\Big\{\frac{1}{|\mu_{m+1}+2\Lambda|^3}+\frac{1}{|\mu_{m+1}-2\Lambda|^3}\Big\}<\infty.
\end{aligned}$$

Similarly,

$$L_2 \leq 48\|B\|^2N^2\|F\|^2_\infty(1+d^3)^2\Big(\frac{1}{|\mu_m+2\Lambda|^3}+\frac{1}{|\mu_m-2\Lambda|^3}\Big)<\infty.$$

As for terms L_3 and L_4, we have

$$\begin{aligned}
L_3 \leq \;& 3N^2\|\nabla F\|^2_\infty\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big(\int_{-\infty}^t\mathrm{e}^{\Lambda|\hat{s}|}\mathrm{e}^{\frac12\mu_{m+1}(t-\hat{s})}|\mathcal{D}^l_rY^N(\hat{s},\cdot)|d\hat{s}\Big)^2dr \\
\leq \;& 12N^2\|\nabla F\|^2_\infty\Big(\frac{1}{|\mu_{m+1}-4\Lambda|}+\frac{1}{|\mu_{m+1}+4\Lambda|}\Big)\int_{-\infty}^t\mathrm{e}^{\frac12\mu_{m+1}(t-\hat{s})}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}^l_rY^N(\hat{s},\cdot)|^2drd\hat{s}
\end{aligned}$$

Note that $\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}^l_rY^N(\hat{s},\cdot)|^2dr$ is nonegative and periodic with period τ, i.e.,

$$\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}^l_rY^N(\hat{s}+\tau,\cdot)|^2dr=\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}^l_rY^N(\hat{s},\theta_\tau\cdot)|^2dr=\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}^l_rY^N(\hat{s},\cdot)|^2dr,$$

where the right equality is true according to Lemma 2.3. Then we have

$$\begin{aligned}\int_{-\infty}^{t} e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l Y^N(\hat{s},\cdot)|^2 dr d\hat{s} &\leq \sup_{s\in[0,\tau)} \mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l Y^N(s,\cdot)|^2 dr \int_{-\infty}^{t} e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}d\hat{s}\\ &\leq \frac{2e^{2\Lambda\tau}}{|\mu_{m+1}|}\sup_{s\in[0,\tau)} e^{-2\Lambda|s|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l Y^N(s,\cdot)|^2 dr.\end{aligned}$$

Thus,

$$L_3 \leq \Big(\frac{24N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}}{|\mu_{m+1}(\mu_{m+1}-4\Lambda)|}+\frac{24N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}}{|\mu_{m+1}(\mu_{m+1}+4\Lambda)|}\Big)\sup_{s\in[0,\tau)} e^{-2\Lambda|s|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l Y^N(s,\cdot)|^2 dr<\infty.$$

Similarly,

$$L_4 \leq \Big(\frac{24N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}}{\mu_m(\mu_m-4\Lambda)}+\frac{24N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}}{\mu_m(\mu_m+4\Lambda)}\Big)\sup_{s\in[0,\tau)} e^{-2\Lambda|s|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l Y^N(s,\cdot)|^2 dr<\infty.$$

(iii) It remains to show that for any $l\in\{1,\cdots,M\}$ and $\delta\in\mathbb{R}$,

$$\sup_{t\in[0,\tau)}\frac{e^{-2\Lambda|t|}}{|\delta|}\int_{\mathbb{R}}\mathbb{E}|\mathcal{D}_{r+\delta}^l\mathcal{M}^N(Y^N)(t,\cdot)-\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr<\infty.$$

In fact the left hand side of the above can be separated into three integrals,

$$\begin{aligned}&\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{e^{-2\Lambda|t|}}{|\delta|}\int_{\mathbb{R}}\mathbb{E}|\mathcal{D}_{r+\delta}^l\mathcal{M}^N(Y^N)(t,\cdot)-\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr\\ =\ &\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{-\infty}^{t-\delta}|\mathcal{D}_{r+\delta}^l\mathcal{M}^N(Y^N)(t,\cdot)-\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr\\ &+\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t}|\mathcal{D}_{r+\delta}^l\mathcal{M}^N(Y^N)(t,\cdot)-\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr\\ &+\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t}^{+\infty}|\mathcal{D}_{r+\delta}^l\mathcal{M}^N(Y^N)(t,\cdot)-\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr\\ :=\ &\hat{K}_1+\hat{K}_2+\hat{K}_3.\end{aligned}\tag{4.9}$$

To estimate $\hat{K}_1$ in (4.9), note when $r\leq t-\delta$, by (4.8) we have

$$\begin{aligned}\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\omega) &= \int_{-\infty}^{r}\mathcal{D}_r^l(\Phi_{t-\hat{s},\hat{s}}^N P^-)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\\ &+\int_{-\infty}^{t}\Phi_{t-\hat{s},\hat{s}}^N P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}_r^l Y^N(\hat{s},\omega)d\hat{s}\\ &-\int_{t}^{+\infty}\Phi_{t-\hat{s},\hat{s}}^N P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}_r^l Y^N(\hat{s},\omega)d\hat{s},\end{aligned}$$

and

$$\mathcal{D}_{r+\delta}^l\mathcal{M}^N(Y^N)(t,\omega) = \int_{-\infty}^{r+\delta}\mathcal{D}_{r+\delta}^l(\Phi_{t-\hat{s},\hat{s}}^N P^-)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}$$

$$
\begin{aligned}
&+\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta}Y^N(\hat{s},\omega)d\hat{s}\\
&-\int_{t}^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta}Y^N(\hat{s},\omega)d\hat{s}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\hat{K}_1 \quad \leq& \sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{3\mathrm{e}^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{-\infty}^{t-\delta}\Big\{\Big|\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(t,\cdot)d\hat{s}\Big|^2\\
&+\Big|\int_{-\infty}^{r+\delta}\mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}-\int_{-\infty}^{r}\mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2\\
&+\Big|\int_{t}^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(t,\cdot)d\hat{s}\Big|^2\Big\}dr\\
:=& \sup_{t\in[0,\tau),\delta\in\mathbb{R}}\sum_{i=1}^{3}Q_i.
\end{aligned}
$$

First note that Q_1 is bounded via measure preserving result in Lemma 2.3,

$$
\begin{aligned}
&Q_1\\
\leq\ & \frac{6N^2\|\nabla F\|_\infty^2}{|\delta|}\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big(\int_{-\infty}^{t}\mathrm{e}^{\Lambda\hat{s}}\mathrm{e}^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|d\hat{s}\Big)^2dr\\
&+\frac{6N^2\|\nabla F\|_\infty^2}{|\delta|}\mathrm{e}^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big(\int_{-\infty}^{t}\mathrm{e}^{-\Lambda\hat{s}}\mathrm{e}^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|d\hat{s}\Big)^2dr\\
\leq\ & \Big(\frac{12N^2\|\nabla F\|_\infty^2}{|\delta||\mu_{m+1}+4\Lambda|}+\frac{12N^2\|\nabla F\|_\infty^2}{|\delta||\mu_{m+1}-4\Lambda|}\Big)\mathbb{E}\int_{\mathbb{R}}\int_{-\infty}^{t}\mathrm{e}^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2d\hat{s}dr
\end{aligned}
$$

Note that $\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr$ is a nonegative periodic function in $C^\Lambda(\mathbb{R})$ with period τ. Thus we have

$$
\begin{aligned}
&\int_{-\infty}^{t}\mathrm{e}^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2drd\hat{s}\\
\leq\ & \sup_{s\in[0,\tau)}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr\int_{-\infty}^{t}\mathrm{e}^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}d\hat{s}\\
\leq\ & \frac{2\mathrm{e}^{2\Lambda\tau}}{|\mu_{m+1}|}\sup_{s\in[0,\tau)}\mathrm{e}^{-2\Lambda|s|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
Q_1 \quad \leq\ & \mathrm{e}^{2\Lambda\tau}\frac{24N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}|}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big)\\
&\cdot\sup_{\hat{s}\in[0,\tau),\delta\in\mathbb{R}}\frac{\mathrm{e}^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr<\infty.
\end{aligned}
$$

Analogously,

$$
Q_3 \quad \leq \quad 24\mathrm{e}^{2\Lambda\tau}N^2\|\nabla F\|_\infty^2\Big\{\frac{1}{\mu_m(\mu_m+4\Lambda)}+\frac{1}{\mu_m(\mu_m-4\Lambda)}\Big\}
$$

$$\cdot \sup_{\hat{s}\in[0,\tau),\delta\in\mathbb{R}} \frac{\mathrm{e}^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}} |(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2 dr<\infty.$$

Secondly, Q_2 can be estimated using (4.4),

$$\begin{aligned}
Q_2 &\le \frac{3\mathrm{e}^{-2\Lambda|t|}\|F\|^2_\infty}{|\delta|}\int_{-\infty}^{t-\delta}\mathbb{E}\Big(\int_r^{r+\delta}\|\mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^-\|d\hat{s}\Big)^2 dr\\
&\le 3N^2\mathrm{e}^{-2\Lambda|t|}\|F\|^2_\infty\|B_l\|(1+2d^3)^2\frac{1}{|\delta|}\int_{-\infty}^{t-\delta}\Big(\int_r^{r+\delta}\mathrm{e}^{\frac12\mu_{m+1}(t-\hat{s})}\mathrm{e}^{\Lambda|\hat{s}|}d\hat{s}\Big)^2 dr\\
&\le 6N^2\|F\|^2_\infty\|B_l\|(1+2d^3)^2\int_{-\infty}^{t-\delta}\mathrm{e}^{(\mu_{m+1}-2\Lambda)(t-\delta-r)}\int_r^{r+\delta}\mathrm{e}^{(\frac12\mu_{m+1}-\Lambda)(r+\delta-\hat{s})}d\hat{s}dr\\
&\quad +6N^2\|F\|^2_\infty\|B_l\|(1+2d^3)^2\int_{-\infty}^{t-\delta}\mathrm{e}^{(\mu_{m+1}+2\Lambda)(t-\delta-r)}\int_r^{r+\delta}\mathrm{e}^{(\frac12\mu_{m+1}+\Lambda)(r+\delta-\hat{s})}d\hat{s}dr\\
&\le 12N^2\|F\|^2_\infty\|B_l\|(1+2d^3)^2\Big\{\frac{1}{(\mu_{m+1}+2\Lambda)^2}+\frac{1}{(\mu_{m+1}-2\Lambda)^2}\Big\}<\infty.
\end{aligned}$$

Thus $\hat{K}_1<\infty$. To consider $\hat{K}_2$ in (4.9), note that when $r\le t\le r+\delta$, the expressions (4.8) gives us

$$\begin{aligned}
\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\omega) &= \int_{-\infty}^{r}\mathcal{D}^l_r(\Phi^N_{t-\hat{s},\hat{s}}P^-)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\\
&\quad +\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_rY^N(\hat{s},\omega)d\hat{s}\\
&\quad -\int_{t}^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_rY^N(\hat{s},\omega)d\hat{s},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\omega) &= \int_{r+\delta}^{+\infty}\mathcal{D}^l_{r+\delta}(\Phi^N_{t-\hat{s},\hat{s}}P^+)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\\
&\quad +\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta}Y^N(\hat{s},\omega)d\hat{s}\\
&\quad -\int_{t}^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta}Y^N(\hat{s},\omega)d\hat{s}.
\end{aligned}$$

Thus

$$\begin{aligned}
\hat{K}_2 &= \sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{\mathrm{e}^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t}|\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot)-\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2dr\\
&\le \sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{4\mathrm{e}^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t}\Big|\int_{-\infty}^{r}\mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2dr\\
&\quad +\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{4\mathrm{e}^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t}\Big|\int_{r+\delta}^{+\infty}\mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2dr\\
&\quad +\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{4\mathrm{e}^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t}\Big|\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2dr
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{4\mathrm{e}^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t}\Big|\int_{t}^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^{+}\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2dr\\
:=&\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\sum_{i=4}^{7}Q_i.
\end{aligned}
$$

But

$$
\begin{aligned}
Q_4 &\le \frac{4\|F\|^2_\infty}{|\delta|}\mathrm{e}^{-2\Lambda|t|}\int_{t-\delta}^{t}\mathbb{E}\Big(\int_{-\infty}^{r}\|\mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^{-}\|d\hat{s}\Big)^2dr\\
&\le 32\|F\|^2_\infty\|B_l\|^2(1+2d^3)\Big(\frac{1}{|\mu_{m+1}-2\Lambda|^2}+\frac{1}{|\mu_{m+1}+2\Lambda|^2}\Big)<\infty.
\end{aligned}
$$

Similarly,

$$
Q_5 \le 32\|F\|^2_\infty\|B_l\|^2(1+2d^3)\Big(\frac{1}{|\mu_{m}-2\Lambda|^2}+\frac{1}{|\mu_{m}+2\Lambda|^2}\Big)<\infty.
$$

Besides, we have by similar calculations as in Q_1 and Q_2,

$$
\begin{aligned}
Q_6 &\le \mathrm{e}^{2\Lambda\tau}\frac{32N^2\|\nabla F\|^2_\infty}{|\mu_{m+1}|}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big)\\
&\quad\cdot\sup_{\hat{s}\in[0,\tau),\delta\in\mathbb{R}}\frac{\mathrm{e}^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr<\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
Q_7 &:= \frac{4\mathrm{e}^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t}\Big|\int_{t}^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^{+}\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2dr\\
&\le 32\mathrm{e}^{2\Lambda\tau}N^2\|\nabla F\|^2_\infty\Big(\frac{1}{|\mu_m(\mu_m+4\Lambda)|}+\frac{1}{|\mu_m(\mu_m-4\Lambda)|}\Big)\\
&\quad\cdot\sup_{\hat{s}\in[0,\tau),\delta\in\mathbb{R}}\frac{\mathrm{e}^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr\\
&< \infty.
\end{aligned}
$$

Now we have shown that $\hat{K}_2<\infty$. To consider $\hat{K}_3$, note that when $r\ge t$, (4.8) gives

$$
\begin{aligned}
\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\omega) &= \int_{r}^{+\infty}\mathcal{D}^l_r(\Phi^N_{t-\hat{s},\hat{s}}P^{+})F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\\
&\quad+\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^{-}\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_rY^N(\hat{s},\omega)d\hat{s}\\
&\quad-\int_{t}^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^{+}\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_rY^N(\hat{s},\omega)d\hat{s},
\end{aligned}
$$

and

$$
\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\omega) = \int_{r+\delta}^{+\infty}\mathcal{D}^l_{r+\delta}(\Phi^N_{t-\hat{s},\hat{s}}P^{+})F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}
$$

$$
\begin{aligned}
&+\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta}Y^N(\hat{s},\omega)d\hat{s}\\
&-\int_{t}^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta}Y^N(\hat{s},\omega)d\hat{s}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\hat{K}_3 &= \sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{\mathrm{e}^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_t^{+\infty}|\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot)-\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2dr\\
&\leq \sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{3\mathrm{e}^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_t^{+\infty}\Big\{\Big|\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2\\
&\quad+\Big|\int_{r+\delta_r}^{+\infty}\mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}-\int_r^{+\infty}\mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2\\
&\quad+\Big|\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2\Big\}dr\\
&= \sup_{t\in\mathbb{R},\delta\in\mathbb{R}}\sum_{i=8}^{10}Q_i.
\end{aligned}
$$

Now it is easy to see that,

$$
\begin{aligned}
Q_8 &\leq \mathrm{e}^{2\Lambda\tau}\frac{24N^2\|\nabla F\|^2_\infty}{|\mu_{m+1}|}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big)\\
&\quad\cdot\sup_{\hat{s}\in[0,\tau),\delta\in\mathbb{R}}\frac{\mathrm{e}^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr<\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
Q_{10} &\leq 24\mathrm{e}^{2\Lambda\tau}N^2\|\nabla F\|^2_\infty\Big(\frac{1}{|\mu_m(\mu_m+4\Lambda)|}+\frac{1}{|\mu_m(\mu_m-4\Lambda)|}\Big)\\
&\quad\cdot\sup_{\hat{s}\in[0,\tau),\delta\in\mathbb{R}}\frac{\mathrm{e}^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr<\infty.
\end{aligned}
$$

Similar to Q_2,

$$
Q_9\leq 24N^2\|F\|^2_\infty\|B_l\|(1+2d^3)^2\Big\{\frac{1}{(\mu_m+2\Lambda)^2}+\frac{1}{(\mu_m-2\Lambda)^2}\Big\}<\infty.
$$

In summary, we have shown that

$$
\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{\mathrm{e}^{-2\Lambda|t|}}{|\delta|}\int_{\mathbb{R}}\mathbb{E}|\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot)-\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2dr<\infty.
$$

Thus we could conclude that $\mathcal{M}^N$ maps $C^\Lambda_\tau(\mathbb{R},\mathcal{D}^{1,2})$ into itself.

Step 2: Now we can prove that for each $N\in\mathbb{N}$, $\mathcal{M}^N(C^\Lambda_\tau(\mathbb{R},\mathcal{D}^{1,2}))|_{[0,\tau)}$ is relatively compact in $C([0,\tau),L^2(\Omega,\mathbb{R}^d))$. Applying Theorem 4.1 and what we have proved in ***Step 1***, we conclude

that that for any sequence $\{\mathcal{M}^N(f_n)\}_{n\in\mathbb{N}} \in C^{\Lambda}_{\tau}(\mathbb{R},\mathcal{D}^{1,2})|_{[0,\tau)}$, there exists a subsequence, still denoted by $\{\mathcal{M}^N(f_n)\}_{n\in\mathbb{N}}$ and $V^N \in C([0,\tau),L^2(\Omega,\mathbb{R}^d))$ such that

$$\sup_{t\in[0,\tau)} \mathbb{E}|\mathcal{M}^N(f_n)(t,\cdot) - V^N(t,\cdot)|^2 \to 0 \tag{4.10}$$

as $n\to\infty$. □

Remark 4.8. *Note that in Theorem 4.1, the relative compactness criterion allows us to apply $L^2(\Omega,\mathbb{R}^d))$-valued functions only on a finite time interval. But we can push it to the whole real line by the random periodicity.*

Proof of Proposition 4.2. We prove for any fixed N, $\mathcal{M}^N(C^{\Lambda}_{\tau}(\mathbb{R},\mathcal{D}^{1,2}))$ is relatively compact in $C^{\Lambda}_{\tau}(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$. Due to the relative compactness in $C([0,\tau),L^2(\Omega,\mathbb{R}^d))$, we are able to find a subsequence, denoted by $\{\mathcal{M}^N(Y^N_{n_j})\}_{j\in\mathbb{N}}$, from an arbitrary sequence $\{\mathcal{M}^N(Y^N_n)\}_{n\in\mathbb{N}}$ such that it will converge to the accumulation point V^N, in the norm shown in Eqn. (4.10). Now define for any $t\in[m\tau,m\tau+\tau)$,

$$V^N(t,\omega) = V^N(t-m\tau,\theta_{m\tau}\omega).$$

By the construction, we can see as $t+\tau\in[(m+1)\tau,(m+2)\tau)$, so

$$V^N(t+\tau,\omega) = V^N(t+\tau-(m+1)\tau,\theta_{(m+1)\tau}\omega) = V^N(t-m\tau,\theta_{m\tau}\theta_\tau\omega) = V^N(t,\theta_\tau\omega).$$

Note that

$$\mathcal{M}^N(Y^N_{n_j})(t,\theta_{m\tau}\omega) = \mathcal{M}^N(Y^N_{n_j})(t+m\tau,\omega).$$

With (4.10), the periodic property of $\mathcal{M}^N(Y^N_{n_j})$, and the probability preserving of $\theta_{m\tau}$, we obtain

$$\begin{aligned}
&\sup_{t\in[m\tau,m\tau+\tau)} \mathrm{e}^{-2\Lambda|t|}\mathbb{E}|\mathcal{M}^N(Y^N_{n_j})(t,\cdot) - V^N(t,\cdot)|^2\\
\le\ &\sup_{t\in[0,\tau)} \mathbb{E}|\mathcal{M}^N(Y^N_{n_j})(t+m\tau,\cdot) - V^N(t+m\tau,\cdot)|^2\\
=\ &\sup_{t\in[0,\tau)} \mathbb{E}|\mathcal{M}^N(Y^N_{n_j})(t,\theta_{m\tau}\cdot) - V^N(t,\theta_{m\tau}\cdot)|^2\\
=\ &\sup_{t\in[0,\tau)} \mathbb{E}|\mathcal{M}^N(Y^N_{n_j})(t,\cdot) - V^N(t,\cdot)|^2 \to 0,
\end{aligned}$$

Thus

$$\sup_{t\in\mathbb{R}} \mathrm{e}^{-2\Lambda|t|}E|\mathcal{M}^N(Y^N_{n_j})(t,\cdot) - V^N(t,\cdot)|^2 \to 0,$$

as $j\to\infty$. Therefore $\mathcal{M}^N(C^{\Lambda}_{\tau}(\mathbb{R},\mathcal{D}^{1,2}))$ is relatively compact in $C^{\Lambda}_{\tau}(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$. □

Theorem 4.9. *Under the same conditions of Proposition 4.2, there exists a $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}$-measurable map $Y:\mathbb{R}\times\Omega\to\mathbb{R}^d$ satisfying Eqn. (3.5) and $Y(t+\tau,\omega) = Y(t,\theta_\tau\omega)$ for any $t\in\mathbb{R}$ and $\omega\in\Omega$.*

Proof. According to Schauder's fixed point theorem, $\mathcal{M}^N$ has a fixed point in $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega, \mathbb{R}^d))$. That is to say there exists a solution $Y^N \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega, \mathbb{R}^d))$ of equation (3.5) such that for any $t \in \mathbb{R}$, $Y^N(t+\tau, \omega) = Y^N(t, \theta_\tau\omega)$. Moreover, $Y^N(t+\tau, \omega) = Y^N(t, \theta_\tau\omega)$.

Recall Ω_N as defined in (3.9). As the random variable

$$\max\left\{\sup_{t\geq 0} \|\Phi(t, \theta_s\omega)P^-\|\mathrm{e}^{-\frac{1}{2}\mu|t|},\ \sup_{t\leq 0} \|\Phi(t, \theta_s\omega)P^+\|\mathrm{e}^{-\frac{1}{2}\mu|t|}\right\}$$

is tempered from above, it is easy to see that

$$\mathbb{P}(\Omega_N) \to 1, \qquad \text{as} \quad N \to \infty.$$

Note also that Ω_N is an increasing sequence of sets. Thus $\cup_N \Omega_N = \hat{\Omega}$ and $\hat{\Omega}$ has the full measure. In fact

$$\hat{\Omega} := \left\{\omega: \sup_{s\in\mathbb{R}} \max\left\{\sup_{t\geq 0} \|\Phi(t, \theta_s\omega)P^-\|\mathrm{e}^{-\frac{1}{2}\mu|t|-\Lambda|s|},\ \sup_{t\leq 0} \|\Phi(t, \theta_s\omega)P^+\|\mathrm{e}^{-\frac{1}{2}\mu|t|-\Lambda|s|}\right\} < \infty\right\}.$$

Therefore it is invariant with respect to θ_s for all $s \in \mathbb{R}$. Now we define

$$\Omega_N^* = \bigcap_{n=-\infty}^{\infty} \theta_{n\tau}^{-1}\Omega_N.$$

Then it is easy to see that Ω_N^* is invariant with respect to $\theta_{n\tau}$ for each n. Besides we have $\Omega_N^* \subset \Omega_{N+1}^*$, which leads to

$$\bigcup_N \Omega_N^* = \bigcup_N \bigcap_{n=-\infty}^{\infty} \theta_{n\tau}^{-1}\Omega_N = \bigcap_{n=-\infty}^{\infty} \theta_{n\tau}^{-1}\left(\bigcup_N \Omega_N\right) = \bigcap_{n=-\infty}^{\infty} \theta_{n\tau}^{-1}\hat{\Omega} = \bigcap_{n=-\infty}^{\infty} \hat{\Omega} = \hat{\Omega},$$

with $\mathbb{P}(\hat{\Omega}) = 1$. Now we can define $Y: \hat{\Omega} \times \mathbb{R} \to \mathbb{R}^d$ as an combinations of Y_N as follows

$$Y := Y_1\chi_{\Omega_1^*} + Y_2\chi_{\Omega_2^*\backslash\Omega_1^*} + \cdots + Y_N\chi_{\Omega_N^*\backslash\Omega_{N-1}^*} + \cdots. \tag{4.11}$$

Thus it is easy to see that Y is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$ measurable and satisfies the following property

$$\begin{aligned}
& Y(t+\tau, \omega) \\
= \ & Y_1(t+\tau, \omega)\chi_{\Omega_1^*}(\omega) + Y_2(t+\tau, \omega)\chi_{\Omega_2^*\backslash\Omega_1^*}(\omega) + \cdots + Y_N(t+\tau, \omega)\chi_{\Omega_N^*\backslash\Omega_{N-1}^*}(\omega) + \cdots \\
= \ & Y_1(t, \theta_\tau\omega)\chi_{\Omega_1^*}(\omega) + Y_2(t, \theta_\tau\omega)\chi_{\Omega_2^*\backslash\Omega_1^*}(\omega) + \cdots + Y_N(t, \theta_\tau\omega)\chi_{\Omega_N^*\backslash\Omega_{N-1}^*}(\omega) + \cdots \\
= \ & Y_1(t, \theta_\tau\omega)\chi_{\Omega_1^*}(\theta_\tau\omega) + Y_2(t, \theta_\tau\omega)\chi_{\Omega_2^*\backslash\Omega_1^*}(\theta_\tau\omega) + \cdots + Y_N(t, \theta_\tau\omega)\chi_{\Omega_N^*\backslash\Omega_{N-1}^*}(\theta_\tau\omega) + \cdots \\
= \ & Y(t, \theta_\tau\omega).
\end{aligned}$$

Moreover Y is a fixed point of $\mathcal{M}$.

We can easily extend Y to the whole Ω as $\mathbb{P}(\hat{\Omega}) = 1$, which is indistinguishable with Y defined in (4.11). □

Remark 4.10. *It is easy to see from (4.11) that $|Y| < \infty$ $\mathbb{P}-a.s.$ Moreover, we don't know whether or not $Y \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega, \mathbb{R}^d))$. However, for each N, $Y^N \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega, \mathbb{R}^d))$. This suggests that $Y \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega_N, \mathbb{R}^d))$ for each N. That is to say that $Y \in C_\tau^\Lambda(\mathbb{R}, L^2_{loc}(\Omega, \mathbb{R}^d))$.*

Now we combine the methods introduced in this section and in [13] to study the following stochastic differential equations

$$du = (Au + F(t,u))dt + \sum_{k=1}^{M} B_k u \circ dW_t^k + \sum_{k=1}^{M} \beta_k(t) dW_t^k, \tag{4.12}$$

and

$$\begin{aligned} \hat{Y}(t,\omega) &= \int_{-\infty}^{t} \Phi(t-s,\theta_s\omega)P^- F(s,\hat{Y}(s,\omega))ds - \int_{t}^{\infty} \Phi(t-s,\theta_s\omega)P^+ F(s,\hat{Y}(s,\omega))ds \\ &\quad + \sum_{k=1}^{M} \int_{-\infty}^{t} \Phi(t-s,\theta_s\omega)P^- \beta_k(s) dW_s^k - \sum_{k=1}^{M} \int_{t}^{\infty} \Phi(t-s,\theta_s\omega)P^+ \beta_k(s) dW_s^k. \end{aligned} \tag{4.13}$$

In fact, this is only a combination of Eqn. (3.1) we considered already in this section and [13]. We include the result here as it is needed in the next section.

Theorem 4.11. *Assume that A, F and B_k satisfy the same conditions as in Proposition 4.2. Let $\beta_k(t) = \beta_k(t+\tau)$ for any $t \in \mathbb{R}$ and there exists a constant R_1 s.t. $\|\beta_k(s_1) - \beta_k(s_2)\|^2 \le R_1|s_1 - s_2|$. Then there exists a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable map $\hat{Y} : \mathbb{R} \times \Omega \to \mathbb{R}^d$ satisfying Eqn. (4.13) and $\hat{Y}(t+\tau,\omega) = \hat{Y}(t,\theta_\tau\omega)$ for any $t \in \mathbb{R}$ and $\omega \in \Omega$.*

Proof. We will adopt the same procedure as in proofs of Proposition 4.2 and Theorem 4.9 and first show the fixed-point existence of the following mapping,

$$\begin{aligned} &\hat{\mathcal{M}}^N(\hat{Y}^N)(t,\omega) \\ = &\int_{-\infty}^{t} \Phi^N(t-s,\theta_s\omega)P^- F(s,\hat{Y}^N(s,\omega))ds - \int_{t}^{\infty} \Phi^N(t-s,\theta_s\omega)P^+ F(s,\hat{Y}^N(s,\omega))ds \\ &+ \sum_{k=1}^{M} \int_{-\infty}^{t} \Phi^N(t-s,\theta_s\omega)P^- \beta_k(s) dW_s^k - \sum_{k=1}^{M} \int_{t}^{\infty} \Phi^N(t-s,\theta_s\omega)P^+ \beta_k(s) dW_s^k. \end{aligned} \tag{4.14}$$

Here the proof differs from Proposition 4.2 in the Malliavin derivative part only, while all the other steps are similar. The Malliavin derivative can be easily computed as follows,

$$\begin{aligned} \mathcal{D}_r^l \hat{\mathcal{M}}^N(\hat{Y}^N)(t,\omega) &= \int_{-\infty}^{r} \chi_{\{r \le t\}}(r) \mathcal{D}_r^l(\Phi_{t-s,s}^N P^-) F(s,\hat{Y}^N(s,\omega))ds \\ &\quad - \int_{r}^{+\infty} \chi_{\{r \ge t\}}(r) \mathcal{D}_r^l(\Phi_{t-s,s}^N P^+) F(s,\hat{Y}^N(s,\omega))ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{t}\Phi^N_{t-s,\hat{s}}P^-\nabla F(s,\hat{Y}^N(s,\omega))\mathcal{D}^l_r\hat{Y}^N(s,\omega)ds\\
&-\int_{t}^{+\infty}\Phi^N_{t-s,s}P^+\nabla F(s,\hat{Y}^N(s,\omega))\mathcal{D}^l_r\hat{Y}^N(s,\omega)ds\\
&+\sum_{k=1}^{M}\int_{-\infty}^{r}\chi_{\{r\le t\}}(r)\mathcal{D}^l_r(\Phi^N_{t-s,s}P^-)\beta_k(s)dW^k_s\\
&-\sum_{k=1}^{M}\int_{r}^{+\infty}\chi_{\{r\ge t\}}(r)\mathcal{D}^l_r(\Phi^N_{t-s,s}P^+)\beta_k(s)dW^k_s\\
&+\chi_{\{r\le t\}}(r)\Phi^N_{t-r,r}P^-\beta_l(r)-\chi_{\{r\ge t\}}(r)\Phi^N_{t-r,r}P^+\beta_l(r).
\end{aligned}
$$

Actually in the above, we only need to take care of the last four terms. The other terms in (4.14) can be dealt with using the same method as in the proof of Proposition 4.2. By Itô isometry, Lemma 4.7 and the property of β, it is easy to see that

$$
\sup_{t\in[0,\tau)}\int_{\mathbb{R}}\mathbb{E}\left|\sum_{k=1}^{M}\int_{r}^{+\infty}\chi_{\{r\ge t\}}(r)\mathcal{D}^l_r(\Phi^N_{t-s,s}P^+)\beta_k(s)dW^k_s\right|^2dr<\infty,
$$

and

$$
\sup_{t\in[0,\tau)}\int_{\mathbb{R}}\mathbb{E}\left|\sum_{k=1}^{M}\int_{-\infty}^{r}\chi_{\{r\le t\}}(r)\mathcal{D}^l_r(\Phi^N_{t-s,s}P^-)\beta_k(s)dW^k_s\right|^2dr<\infty,
$$

Besides, by the estimate (3.4), we can show for each $l\in\{1,2,\cdots,M\}$,

$$
\begin{aligned}
\sup_{t\in[0,\tau)}\int_{\mathbb{R}}\mathbb{E}\left|\chi_{\{r\le t\}}(r)\Phi^N_{t-r,r}P^-\beta_l(r)\right|^2dr &< \infty,\\
\sup_{t\in[0,\tau)}\int_{\mathbb{R}}\mathbb{E}\left|\chi_{\{r\ge t\}}(r)\Phi^N_{t-r,r}P^+\beta_l(r)\right|^2dr &< \infty.
\end{aligned}
$$

Also we can show for each $l\in\{1,2,\cdots,M\}$,

$$
\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{1}{|\delta|}\mathbb{E}\int_{\mathbb{R}}\left|\mathcal{D}^l_{r+\delta}\hat{M}(\hat{Y}^N)(t,\omega)-\mathcal{D}^l_r\hat{M}(\hat{Y}^N)(t,\omega)\right|^2dr<\infty.
$$

To achieve this, we only need to check the following terms,

$$
\begin{aligned}
&\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{1}{|\delta|}\mathbb{E}\int_{\mathbb{R}}\left|\chi_{\{r+\delta\le t\}}(r+\delta)\Phi^N_{t-(r+\delta),(r++\delta)}P^-\beta_l(r+\delta)-\chi_{\{r\le t\}}(r)\Phi^N_{t-r,r}P^-\beta_l(r)\right|^2dr\\
\le\ &\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{3}{|\delta|}\mathbb{E}\int_{t-\delta}^{t}\left|\Phi^N_{t-(r+\delta),(r++\delta)}P^-\beta_l(r+\delta)\right|^2dr\\
&+\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{3}{|\delta|}\mathbb{E}\int_{-\infty}^{t}\left|(\Phi^N_{t-(r+\delta),(r++\delta)}P^--\Phi^N_{t-r,r}P^-)\beta_l(r+\delta)\right|^2dr\\
&+\sup_{t\in[0,\tau),\delta\in\mathbb{R}}\frac{3}{|\delta|}\mathbb{E}\int_{-\infty}^{t}\left|\Phi^N_{t-(r+\delta),(r++\delta)}P^-(\beta_l(r+\delta)-\beta_l(r))\right|^2dr
\end{aligned}
$$

$$=: \sum_{i=1}^{3} A_i < \infty,$$

and by Lemma 4.7 for each $k \in \{1, 2, \cdots, M\}$,

$$\begin{aligned}
& \frac{1}{|\delta|}\mathbb{E}\int_{\mathbb{R}}\Big|\int_{-\infty}^{r+\delta}\chi_{\{r+\delta\le t\}}(r)\mathcal{D}^l_{r+\delta}(\Phi^N_{t-s,s}P^-)\beta_k(s)dW^k_s \\
& \qquad -\int_{-\infty}^{r}\chi_{\{r\le t\}}(r)\mathcal{D}^l_r(\Phi^N_{t-s,s}P^-)\beta_k(s)dW^k_s\Big|^2 dr \\
\le\ & \frac{2}{|\delta|}\left(\mathbb{E}\int_{-\infty}^{t-\delta}\Big|\int_{r}^{r+\delta}\mathcal{D}^l_{r+\delta}(\Phi^N_{t-s,s}P^-)\beta_k(s)dW^k_s\Big|^2 dr\right. \\
& \qquad \left.+\mathbb{E}\int_{t-\delta}^{t}\Big|\int_{r+\delta}^{\infty}\mathcal{D}^l_{r+\delta}(\Phi^N_{t-s,s}P^-)\beta_k(s)dW^k_s\Big|^2 dr\right) \\
=:\ & B_1 + B_2 < \infty.
\end{aligned}$$

The boundedness of A_1 can be derived from the estimate of Φ and boundedness of β, and A_2 by the same way as we dealt with T_1 in the proof of Lemma 4.4. As for A_3, the Lipschitz condition of β works. The boundedness of B_1 and B_2 can be done through the Itô isometry, Lemma 4.6 and boundedness of β.

Then by the same argument as in the proof of Proposition 4.2, we can prove the existence of fixed point for Eqn. (4.14). Using the same "measurable glue" method in the proof of Theorem 4.9 one can obtain a measurable solution Y of Eqn. (4.13) satisfying $\hat{Y}(t+\tau,\omega) = \hat{Y}(t,\theta_\tau\omega)$ for any $t \in \mathbb{R}$ and $\omega \in \Omega$. □

The existence of random periodic solution results obtained in Theorems 4.9 and 4.11, together with the "equivalence" of random periodic solutions and periodic measure obtained in [15], implies the existence of periodic measure with respect to the skew product of the random dynamical system and metric dynamical system. For this, define $\mu : \mathbb{R}\times\Omega\times\mathcal{B}(\mathbb{R}^d) \to [0,1]$ by

$$(\mu_s)_\omega = \delta_{Y(s,\theta(-s)\omega)}. \tag{4.15}$$

Then

$$(\mu_{s+\tau})_\omega = (\mu_s)_\omega$$

and

$$u(t+s,s,\theta(-s)\omega)(\mu_s)_\omega = (\mu_{t+s})_{\theta(s)\omega}.$$

Define the product space $\hat{\Omega} = \Omega\times\mathbb{R}^d$ with σ-field $\hat{\mathcal{F}} = \mathcal{F}\otimes\mathcal{B}(\mathbb{R}^d)$ and the skew product $\Theta : \Delta\times\hat{\Omega} \to \hat{\Omega}$ by

$$\hat{\Theta}(t+s,s)(\omega,x) = (\theta(t)\omega, u(t+s,s,\theta(-s)\omega)x).$$

Then for any $t_1, t_2 \in \mathbb{R}^+$, $s \in \mathbb{R}$,

$$\hat{\Theta}(t_2+t_1+s, t_1+s)\hat{\Theta}(t_1+s,s) = \hat{\Theta}(t_1+t_2+s,s).$$

Set $\mu_s : \hat{\mathcal{F}} \to [0,1]$ has

$$\mu_s(dx, d\omega) = (\mu_s)_\omega(dx) \times \mathbb{P}(d\omega). \tag{4.16}$$

Then $\mu_{s+\tau} = \mu_\tau$, and

$$\hat{\Theta}(t+s, s)\mu_s = \mu_{t+s}, \text{ for any } t \in \mathbb{R}^+, s \in \mathbb{R}.$$

Thus μ_s, $s \in \mathbb{R}$ is a periodic measure of the skew product $\hat{\Theta}$. We omit the full details here, see [15].

5 Applications and examples

First we consider a simple example of stochastic differential equations with time periodic coefficients.

Example 5.1. *Consider*

$$dX(t) = -X(t)dt + c\cos(t)dt + 10\sin(t)dW_t. \tag{5.1}$$

Here c is a constant and $W(t)$ is a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Applying the result in [13] or Theorem 4.11 in this paper, we can assert that Eqn. (5.1) has a random periodic solution. In fact, according to the equivalence theorem (Theorem 2.5), the random periodic solution of (5.1) can be written explicitly as follows

$$\begin{aligned} Y(t) &= \int_{-\infty}^{t} \mathrm{e}^{-t+s} c\cos(s)ds + 10\int_{-\infty}^{t} \mathrm{e}^{-t+s}\sin(s)dW_s \\ &= \frac{c}{2}(\cos(t) + \sin(t)) + 10\int_{-\infty}^{t} \mathrm{e}^{-t+s}\sin(s)dW_s. \end{aligned}$$

Actually it can be verified by direct simple calculations that $Y(t)$ satisfies definition 1.1. When $c = 0$, the solution of Eqn. (5.1) is a Ornstein-Uhlenbeck process with white noise of periodic coefficient. Similar to the case of time independent case that the dynamical system generated by the Orsnstein-Uhlenbeck process has a stationary path and an invariant measure, the example suggests that the semiflow generated by the time periodic Orsnstein-Uhlenbeck process has a random periodic path and a periodic measure.

To study a smilar equation with multiplicative noise, let us consider,

$$dX(t) = -X(t)dt + \cos(t)dt + 10X(t) \circ dW_t. \tag{5.2}$$

Theorem 4.11 implies the existence of random periodic solutions to Eqn. (5.2). According to Theorem 3.4, the random periodic solution of Eqn. (5.2) is give explicitly by

$$Y(t) = \int_{-\infty}^{t} \mathrm{e}^{-(t-s)+10(W(t)-W(s))}\cos(s)ds.$$

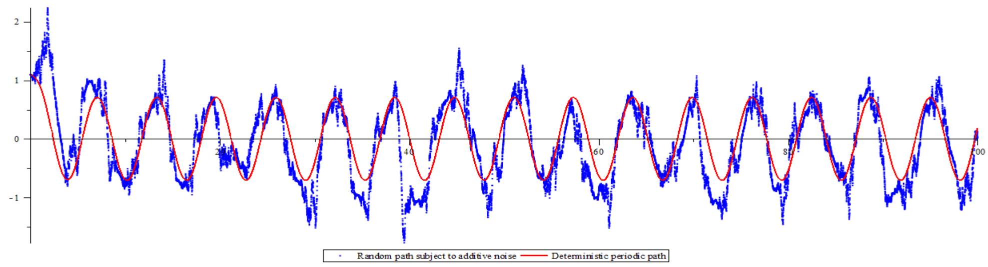

Figure 1: Random trajectory subject to additive noise

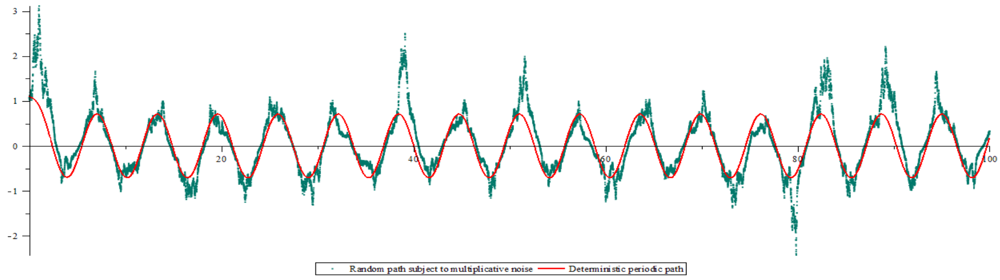

Figure 2: Random trajectory subject to multiplicative noise

The numerical simulations of Eqn. (5.1) (taking $c = 1$) and Eqn. (5.2) displayed by Fig.1 and Fig.2 demonstrate how the random periodic solutions fluctuate around the deterministic periodic solution of the noiseless ordinary differential equation $\frac{d}{dt}X(t) = -X(t) + \cos(t)$ in the additive noise case and the multiplicative noise case respectively.

Secondly, we apply the results of last section to study the following stochastic differential equations on R^d

$$dx = (Ax + f(x))dt + \sum_{k=1}^{M} B_k x \circ dW_t^k + \sum_{k=1}^{M} \gamma_k dW_t^k. \tag{5.3}$$

As some B_k and γ_k can be zero, so this equation includes the case when the multiplicative noise and additive noise are independent, though in both the additive and multiplicative noise terms we use the same multidimensional Brownian motions. It is easy to see that under the conditions of Theorem 4.11, Eqn. (5.3) generates a cocycle random dynamical system $\Psi : R^+ \times \Omega \times R^d \to R^d$ such that for all $\omega \in \Omega$, (c.f. [2])

$$\Psi(t, \theta_s\omega) \circ \Psi(s, \omega) = \Psi(t + s, \omega), \text{for all } t, s \geq 0. \tag{5.4}$$

Here W_k and θ are the same as before. In this case the skew product $\bar{\Theta}$ is defined as $\bar{\Theta}(t)(\omega, x) = (\theta(t)\omega, \Psi(t, \omega)x)$.

Theorem 5.2. *Let A, B_k satisfy the same conditions as in Proposition 4.2 and the function $f \in C^3$ be uniformly bounded with bounded first order derivatives. Assume the deterministic*

system

$$\frac{dx}{dt} = Ax + f(x) \tag{5.5}$$

has a periodic solution z with period $\tau > 0$ and $z(t)$ is C^3 in t. Then Eqn. (5.3) has a random periodic solution of period τ i.e. there exists $Y : R \times \Omega \to R^d$ such that for any $t \in R^+, s \in R$,

$$\Psi(t, \theta(s)\omega)Y(s, \omega) = Y(t + s, \omega), \quad Y(s + \tau, \omega) = Y(s, \theta_\tau \omega). \tag{5.6}$$

Moreover, the measure μ_s given in (4.16) is a periodic measure of $\bar{\Theta}$, and $\bar{\mu} = \frac{1}{\tau}\int_0^\tau \mu_s ds$ is an invariant measure whose random factorisation has the support $\{Y(s, \theta(-s)\omega) : \ 0 \le s < \tau\}$ which is a closed curve.

Proof. Let

$$u(t, s, \omega)u_0 = \Psi(t - s, \theta(s)\omega)(u_0 + z(s)) - z(t), \quad t \ge s$$

with Ψ satisfying Eqn. (5.3). Then $u(t)$ (in short for $u(t, s)$) satisfies

$$du(t) = (Au(t) + f(u(t) + z(t)) - f(z(t)))dt + \sum_{k=1}^{M} B_k u(t) \circ dW_t^k + \sum_{k=1}^{M}(B_k z(t) + \gamma_k)dW_t^k,$$

with $u(s) = u_0$. The above equation has a random periodic solution $\hat{Y}$ by Theorem 4.11, so Eqn. (5.3) has a random periodic solution $Y(t, \omega) = \hat{Y}(t, \omega) + z(t)$. In fact for any $t \ge 0$,

$$\begin{aligned} \Psi(t, \theta(s)\omega)Y(s, \omega) &= u(t, s, \omega)(Y(s, \omega) - z(s)) + z(t + s) \\ &= u(t, s, \omega)\hat{Y}(s, \omega) + z(t + s) \\ &= \hat{Y}(t + s, \omega) + z(t + s) = Y(t + s, \omega), \end{aligned}$$

and the random periodicity of Y is obvious. The last claim follows from the existence of a random periodic solution and the result on periodic and invariant measures in [15]. □

In the above theorem, the main assumption is that the deterministic system (when noise is switched off) has a periodic solution, the other assumptions are very mild. Many ordinary differential equations modeling real world problems arising from biology, chemistry, chemical engineering, climate dynamics, economics etc. have periodic solutions. Here we can make the function f bounded by a smooth truncation procedure outside a sufficiently large ball mentioned in [14] if necessary without changing their local dynamical behavior. Therefore Theorem 5.2 gives the existence of random periodic solutions of stochastic differential equations arising from many real world applications.

As a simple, but typical example, we consider a random dynamical system generated by a random perturbation to the following ordinary differential equation in $\mathbb{R}^2$,

$$\begin{cases} dy_1(t) = -y_2(t)dt - y_1(t)dt + y_1(t)(2 - y_1^2(t) - y_2^2(t))\phi(y_1(t), y_2(t))dt, \\ dy_2(t) = y_1(t)dt - y_2(t)dt + y_2(t)(2 - y_1^2(t) - y_2^2(t))\phi(y_1(t), y_2(t))dt, \end{cases} \tag{5.7}$$

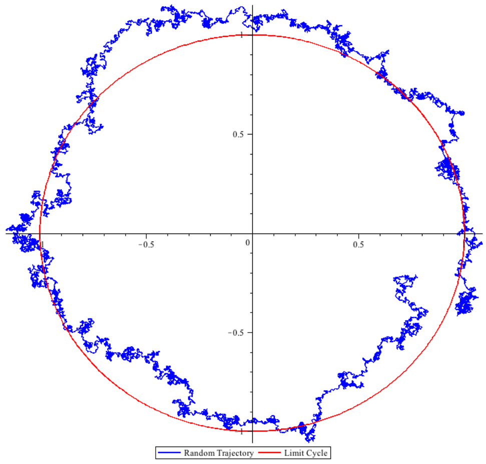

Figure 3: Random trajectory subject to additive noise

where ϕ is a smooth function such that

$$\phi(y_1, y_2) = \begin{cases} 1, \text{ when } y_1^2 + y_2^2 \leq 2^{100}, \\ 0, \text{ when } y_1^2 + y_2^2 \geq 2^{101}. \end{cases} \tag{5.8}$$

It is not hard to see that the limit cycle of this system is $y_1^2 + y_2^2 = 1$.

Example 5.3. *Consider the corresponding stochastic differential equation with additive noise,*

$$\begin{cases} dy_1(t) = -y_2(t)dt - y_1(t)dt + y_1(t)(2 - y_1^2(t) - y_2^2(t))\phi(y_1(t), y_2(t))dt + 10dW_t^1, \\ dy_2(t) = y_1(t)dt - y_2(t)dt + y_2(t)(2 - y_1^2(t) - y_2^2(t))\phi(y_1(t), y_2(t))dt + 10dW_t^2. \end{cases} \tag{5.9}$$

We can apply the result in [14] or Theorem 5.2 to assert that Eqn. (5.9) has a random periodic solution. Fig.3 illustrates a numerical simulation of a sample path of the random periodic solution. Fig.4 illustrates the formulation of the invariant measure. We start at $t = 0$ with uniform distribution on $[-2, 2] \times [-2, 2]$. Subject to the same random perturbations, all those points begin to move towards a random closed curve simultaneously. In the long run, they evolve into a random closed curve, which represents the support of the invariant measure with respect the skew product dynamical system as described in Theorem 5.2. The closed curve moves randomly, its law is a periodic measure of the corresponding Markovian semigroup.

Now we consider the random perturbation of (5.7) subject to multiplicative linear noise,

$$\begin{cases} dy_1(t) = -y_2(t)dt - y_1(t)dt + y_1(t)(2 - y_1^2(t) - y_2^2(t))\phi(y_1(t), y_2(t))dt + 10y_1 \circ dW_t^1, \\ dy_2(t) = y_1(t)dt - y_2(t)dt + y_2(t)(2 - y_1^2(t) - y_2^2(t))\phi(y_1(t), y_2(t))dt + 10y_2 \circ dW_t^2. \end{cases} \tag{5.10}$$

Note the commutativity assumption in Theorem 4.11 is not satisfied. But following Remark 4.3, we can still use the previous result to conclude the Eqn. (5.10) has a random periodic solution. A sample path is given by a numerical simulation in Fig.5. Similar phenomena as the formation of an invariant measure as demonstrated for Eqn. (5.9) in Fig.4 can be obtained. The detail is omitted here.

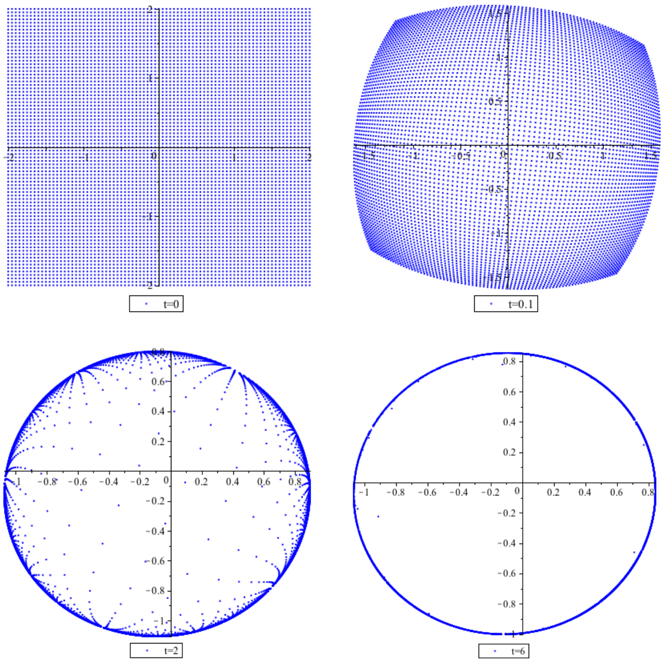

Figure 4: Random periodic evolution

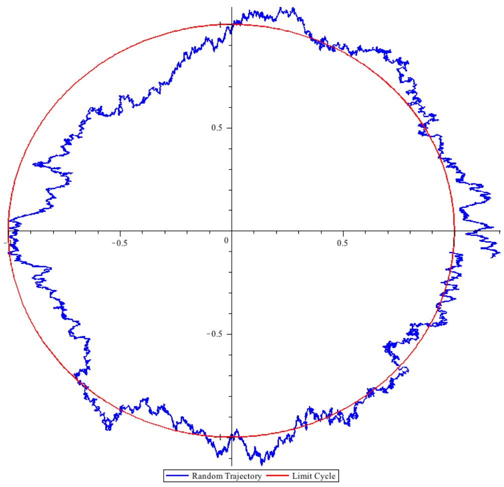

Figure 5: Random Trajectory subject to multiplicative linear noise

Appendix

Proof of Lemma 3.2. (i). It is easy to show that Φ satisfies the condition $\sup\limits_{0\le t\le 1} \log^+ \|\Phi(t,\omega)^{\pm 1}\| \in L^1(\Omega)$. So the MET theorem ensures the existence of the random Oseledets splitting

$$\mathbb{R}^d = E_p(\omega) \oplus E_{p-1}(\omega) \oplus \cdots \oplus E_{m+1}(\omega) \cdots \oplus E_1(\omega),$$

and the corresponding random projections $P^{\pm}(\omega)$. But if we consider the forward filtration and $\lim_{t\to\infty}(\Phi(t,\omega)^*\Phi(t,\omega))^{1/2t} := \Psi(\omega)$, the mutually commutative property of A, A^*, B_k, and B_k^* leads to

$$\Psi(\omega) = \lim_{t\to\infty} \exp\left\{\frac{1}{2}(A+A^*) + \sum_{k=1}^{M} \frac{(B_k + B_k^*)W_t^k}{2t}\right\} = \exp\left\{\frac{A+A^*}{2}\right\}.$$

Note $e^{\mu_p} < e^{\mu_{p-1}} \cdots < e^{\mu_{m+1}} < 1 < e^{\mu_m} < \cdots < e^{\mu_1}$ are the eigenvalues of $\exp\left\{\frac{A+A^*}{2}\right\}$, and $U_p, \cdots, U_1$ are still the corresponding orthogonal eigenspaces, with multiplicity $d_i := \dim U_i$. Define $V_{p+1} := \{0\}$, and for $1 \le i \le p$, $i \in \mathbb{N}$,

$$V_i := U_p \oplus U_{p-1} \oplus \cdots \oplus U_i. \tag{5.11}$$

Therefore

$$V_p \subset V_{p-1} \subset \cdots \subset V_i \subset \cdots \subset V_1 = \mathbb{R}^d \tag{5.12}$$

defines a forward filtration.

Now we consider the backward filtration and $\lim_{t\to\infty}(\Phi(-t,\omega)^*\Phi(-t,\omega))^{1/2t} := \tilde{\Psi}(\omega)$. Note

$$\tilde{\Psi}(\omega) = \lim_{t\to\infty} \exp\left\{-\frac{1}{2}(A+A^*) - \sum_{k=1}^{M} \frac{(B_k + B_k^*)W_{-t}^k}{2t}\right\} = \exp\left\{-\frac{A+A^*}{2}\right\}.$$

Let $\tilde{\mu}_i = -\mu_{p+1-i}$. Then $\tilde{\mu}_p < \tilde{\mu}_{p-1} \cdots < \tilde{\mu}_{p+1-m} < 0 < \tilde{\mu}_{p-m} < \cdots < \tilde{\mu}_1$ are the eigenvalues of $-\frac{A+A^*}{2}$. Let $\tilde{U}_p, \cdots, \tilde{U}_1$ be the corresponding eigenspaces, with multiplicity $\tilde{d}_i := \dim \tilde{U}_i$. Then $\tilde{U}_i = U_{p+1-i}$. Define $\tilde{V}_{p+1} := \{0\}$, and for $1 \le i \le p$, $i \in \mathbb{N}$,

$$\tilde{V}_i := \tilde{U}_p \oplus \tilde{U}_{p-1} \oplus \cdots \oplus \tilde{U}_i = U_1 \oplus U_2 \oplus \cdots \oplus U_{p+1-i}. \tag{5.13}$$

Therefore

$$\tilde{V}_p \subset \tilde{V}_{p-1} \subset \cdots \subset \tilde{V}_i \subset \cdots \subset \tilde{V}_1 = \mathbb{R}^d \tag{5.14}$$

defines the backward filtration. Then we can construct the space E_i as the intersection of certain spaces from the forward filtration (5.12) and the backward filtration (5.14),

$$E_i := V_i \cap \tilde{V}_{p+1-i} = U_i. \tag{5.15}$$

Thus the Lyapunov exponents of Φ depend on $\frac{1}{2}(A+A^*)$ only. This implies that the Oseledets spaces are non-random and so are the corresponding projections $P^{\pm}$.

(ii). Note when $t \le 0$,

$$\begin{aligned}
\|\Phi(t,\omega)P^+\| &= \|P^{+*}\Phi(t,\omega)^*\Phi(t,\omega)P^+\|^{1/2} \\
&= \left\|P^{+*}\exp\left\{(A+A^*)t+\textstyle\sum_{k=1}^M(B_k+B_k^*)W_t^k\right\}P^+\right\|^{1/2} \\
&\le \left\|P^{+*}\exp\left\{(A+A^*)t\right\}P^+\right\|^{1/2}\left\|\exp\left\{\textstyle\sum_{k=1}^M(B_k+B_k^*)W_t^k\right\}\right\|^{1/2} \\
&\le \left\|\exp\left\{\frac{1}{2}(A+A^*)t\right\}P^+\right\|\exp\left\{\frac{1}{2}\textstyle\sum_{k=1}^M\|B_k+B_k^*\||W_t^k|\right\} \\
&\le e^{\frac{1}{2}\mu t+\sum_{k=1}^M\|B\||W_t^k|}e^{\frac{1}{2}\mu_m t},
\end{aligned}$$

where $\|B\| := \frac{1}{2}\max_{k\in\{1,2,\cdots,M\}}\|B_k+B_k^*\|$, $\mu := \min\{-\mu_{m+1},\mu_m\} > 0$. Define

$$C(\omega) := \sup_{t\in\mathbb{R}} C(t,\omega) := \sup_{t\in\mathbb{R}} \mathrm{e}^{-\frac{1}{2}\mu|t|+\sum_{k=1}^M\|B\||W_t^k|} \ge 1. \tag{5.16}$$

Now it suffices to check that $C(\omega)$ is tempered from above. Similarly as in [10], using $|W(t+s)| \le C_{\delta,\omega}+|s|^\delta+|t|^\delta \quad \mathbb{P}-a.s.$ for some $\frac{1}{2}<\delta<1$, from the iterated logarithm law of Brownian motion, we have

$$\begin{aligned}
\lim_{s\to\pm\infty}\frac{1}{|s|}\log^+\sup_{t\in\mathbb{R}}C(t,\theta_s\omega) &= \lim_{s\to\pm\infty}\frac{1}{|s|}\log\sup_{t\in\mathbb{R}}C(t,\theta_s\omega) \\
&= \lim_{s\to\pm\infty}\frac{1}{|s|}\sup_{t\in\mathbb{R}}\log\mathrm{e}^{-\frac{1}{2}\mu|t|+\sum_{k=1}^M\|B\||\theta_sW_t^k|} \\
&\le \lim_{s\to\pm\infty}\frac{1}{|s|}\sup_{t\in\mathbb{R}}\left(-\frac{1}{2}\mu|t|+\sum_{k=1}^M\|B\||W_{t+s}^k|\right)+\lim_{s\to\pm\infty}\sum_{k=1}^M\|B\|\frac{|W_s^k|}{|s|} \\
&\le \lim_{s\to\pm\infty}\frac{1}{|s|}\sup_{t\in\mathbb{R}}\left(-\frac{1}{2}\mu|t|+M\|B\||t|^\delta\right) \\
&\quad +\sup_{t\in\mathbb{R}}\lim_{s\to\pm\infty}M\|B\|\frac{|s|^\delta}{|s|}+\sup_{t\in\mathbb{R}}\lim_{s\to\pm\infty}\frac{M\|B\||C_{\delta,\omega}|}{|s|} \\
&= \lim_{s\to\pm\infty}\frac{1}{|s|}\sup_{t\in\mathbb{R}}\left(-\frac{1}{2}\mu|t|+M\|B\||t|^\delta\right) = 0, \quad \mathbb{P}-a.s.,
\end{aligned}$$

where the last inequality holds due to the fact that $\sup_{t\in\mathbb{R}}(-\frac{1}{2}\mu|t|+M\|B\||t|^\delta) < \infty$. This together with the fact that

$$\lim_{s\to\pm\infty}\frac{1}{|s|}\log\sup_{t\in\mathbb{R}}C(t,\theta_s\omega) \ge 0,$$

leads to that $C(\omega)$ is a tempered random variable. Similar argument applies to $\Phi(t,\theta_s\omega)P^-$. Finally by definition of random variable tempered from above, we can easily conclude that $\Phi(t,\theta_s\omega)P^-$ and $\Phi(t,\theta_s\omega)P^+$ satisfy (3.4). □

Proof of Corollary 3.3. We consider P^- case only. The estimation for P^+ case can be derived analogously. From Eqn. (3.2) and the definition of P^-, it is natural to express the projection

$\Phi P^- : \mathbb{R} \times \Omega \to \mathcal{L}(\mathbb{R}^d, E^-)$ as follows,

$$\begin{cases} d\Phi(t,\omega)P^- = A\Phi(t)P^-\,dt + \sum_{k=1}^{M} B_k\Phi(t)P^- \circ dW_t^k, \\ \Phi(0,\omega)P^- = P^- \in \mathcal{L}(\mathbb{R}^d, E^-). \end{cases}$$

Then for any $t, \hat{s} \in \mathbb{R}$, by the ergodic property of θ and Holder's inequality we have that

$$\begin{aligned} & \mathbb{E}\|P^- - \Phi(t,\theta_{\hat{s}}\cdot)P^-\|^2 \\ = \;& \mathbb{E}\Big\| \int_0^t \Big(A + \frac{1}{2}\sum_{k=1}^{M} B_k^2\Big)\Phi_{\hat{h},\hat{s}}P^- d\hat{h} + \sum_{k=1}^{M}\int_0^t B_k\Phi_{\hat{h},\hat{s}}P^- dW^k_{\hat{h}+\hat{s}}\Big\|^2 \\ \leq \;& (M+1)\Big\|A + \frac{1}{2}\sum_{k=1}^{M} B_k^2\Big\|^2 |t| \int_0^t \mathbb{E}\|\Phi_{\hat{h}}P^-\|^2 d\hat{h} + (M+1)\sum_{k=1}^{M}\|B_k\|^2 \int_{\hat{s}}^{t+\hat{s}} \mathbb{E}\|\Phi_{\hat{h}-\hat{s}}P^-\|^2 d\hat{h} \\ \leq \;& 2^M(M+1)\Big(2\|A\|^2|t| + \Big(\textstyle\sum_{k=1}^{M}\|B_k\|^2\Big)^2|t| + \sum_{k=1}^{M}\|B_k\|^2\Big)\mathrm{e}^{2\|A\||t| + 2M\|B\|^2|t|}|t|, \end{aligned}$$

where

$$\begin{aligned} \int_0^t \mathbb{E}\|\Phi_{\hat{h}}P^-\|^2 d\hat{h} \;&\leq\; \int_0^t \mathbb{E}\|\mathrm{e}^{A\hat{h} + \sum_{k=1}^{M} B^k W^k_{\hat{h}}}\|^2 d\hat{h} \leq \int_0^t \mathrm{e}^{2\|A\|\hat{h}} \prod_{k=1}^{M} \mathbb{E}\mathrm{e}^{2\|B\||W^k_{\hat{h}}|} d\hat{h} \\ &=\; 2^M \int_0^t \mathrm{e}^{2\|A\|\hat{h}}\mathrm{e}^{2M\|B\|^2\hat{h}} d\hat{h} \leq 2^M \mathrm{e}^{2\|A\||t| + 2M\|B\|^2|t|}|t|. \end{aligned}$$

Finally the last inequality can be easily drawn from above. □

Proof of Theorem 4.1. According to the generalized Arzelà−Ascoli lemma (c.f. [19]), it suffices to check the uniform equicontinuity and pointwise relative compactness of v_n. First we claim that $\{v_n(t,\cdot),\ n \in \mathbb{N}\}$ is relatively compact in $L^2(\Omega)$ for any fixed t. To achieve this, we decompose v_n as Wiener-Itô chaos expansions (c.f.[24]),

$$v_n(t,\omega) = \sum_{m=0}^{\infty} I_m(f_n^m(\cdot,t))(\omega), \tag{5.17}$$

where $f_n^m(\cdot,t)$ are symmetric elements in $L^2(\mathbb{R}^m)$ for each $m \geq 0$ and each $t \in [a,b]$. By the similar argument in the proof of Theorem 1 in [3], the relative compactness of $\{v_n\}_{n\in\mathbb{N}}$ is reduced to the relative compactness of $\{f_n^m\}_{n\in\mathbb{N}}$ for each finite $m \in \mathbb{N}$.

When $m = 0$, $f_n^0(t) = \mathbb{E}v_n(t)$, and for any $t_1, t_2 \in [a,b]$, hypotheses (i) and (ii) imply the uniform boundedness of f_n^0,

$$\sup_n \sup_{t\in[a,b]} |f_n^0(t)| \leq \sup_n \sup_{t\in[a,b]} \sqrt{\mathbb{E}|v_n(t)|^2} \leq \sup_{n\in N} \sup_{t\in[a,b]} ||v_n(\cdot,t)||_{1,2} < \infty.$$

Besides, applying Jensen's inequality gives the uniform equicontinuity of f_n^0,

$$\sup_n |f_n^0(t_1) - f_n^0(t_2)| \;=\; \sup_n |\mathbb{E}(v_n(t_1) - v_n(t_2))| \leq \sup_n \mathbb{E}|v_n(t_1) - v_n(t_2)|$$

$$\leq \sup_n \sqrt{\mathbb{E}|v_n(t_1)-v_n(t_2)|^2} \leq \sqrt{C|t_1-t_2|}.$$

So $\{f_n^0\}_{n=1}^\infty$ is relatively compact in $C([a,b])$ according to the classical Arzelà-Ascoli lemma.

Using a similar argument as in the proof of Theorem 2 in [3] for each $m \geq 1$ with the general relative compactness criterion (c.f.Theorem 2.32 in [1]), we claim that $\{f_n^m(\cdot,t)\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(\mathbb{R}^m)$ for each fixed t. To see this, let $h=(h_1,\ldots,h_m)\in\mathbb{R}^m$. It holds that

$$\begin{aligned}
&\|\tau_h f_n^m - f_n^m\|^2_{L^2(\mathbb{R}^m)}\\
=\ & \int_{\mathbb{R}^m} |f_n^m(t,t_1+h_1,\cdots,t_m+h_m)-f_n^m(t,t_1,\cdots,t_m)|^2 dt_1\cdots dt_m\\
\leq\ & C\sum_{i=1}^m \int_{\mathbb{R}^m} |f_n^m(t,t_1,\cdots,t_{i-1},t_i+h_i,t_{i+1}+h_{i+1},\cdots,t_m+h_m)\\
&\qquad\qquad -f_n^m(t,t_1,\cdots,t_{i-1},t_i,t_{i+1}+h_{i+1},\cdots,t_m+h_m)|^2 dt_1\cdots dt_m\\
=\ & C\sum_{i=1}^m \int_{\mathbb{R}} \|f_n^m(t,\ldots,t_i+h_i,\ldots)-f_n^m(t,\ldots,t_i,\ldots)\|^2_{L^2(\mathbb{R}^{m-1})} dt_i\\
=\ & \frac{C}{(m-1)!}\sum_{i=1}^m \int_{\mathbb{R}} \mathbb{E}|I_{m-1}(f_n^m(t,\cdots,t_i+h_i,\cdots)-f_n^m(t,\cdots,t_i,\cdots))|^2 dt_i\\
\leq\ & \frac{C}{mm!}\sum_{i=1}^m \int_{\mathbb{R}} \mathbb{E}\Big|\sum_{m\geq 1} mI_{m-1}\big(f_n^m(t,\cdots,t_i+h_i,\cdots)-f_n^m(t,\cdots,t_i,\cdots)\big)\Big|^2 dt_i\\
\leq\ & \frac{C}{m!}\int_{\mathbb{R}} \mathbb{E}\Big|\mathcal{D}_{r+h_1}v_n(t)-\mathcal{D}_r v_n(t)\Big|^2 dr \leq C|h_1|,
\end{aligned}$$

where C is a constant depending on m. Moreover, for any $\epsilon>0$, there exists $[\alpha,\beta]\subset\mathbb{R}$ such that

$$\int_{\mathbb{R}\setminus[\alpha,\beta]} \mathbb{E}|\mathcal{D}_r v_n(t)|^2 dr < \epsilon.$$

Let $G=[\alpha,\beta]\times[-1,1]^{d-1}$. Then we have

$$\begin{aligned}
\int_{R^m\setminus\bar{G}} |f_n^m(t,t_1,\cdots,t_m)|^2 dt_1\cdots dt_m \ &\leq\ C\int_{\mathbb{R}\setminus[\alpha,\beta]} \|f_n^m(t,r,t_2,\cdots,t_m)\|^2_{L^2(\mathbb{R}^{m-1})} dr\\
&\leq\ \frac{C}{m!}\int_{\mathbb{R}\setminus[\alpha,\beta]} \mathbb{E}\Big|\sum_{m\geq 1} mI_{m-1}(f_n^m(t,r,\cdot))\Big|^2 dr\\
&\leq\ \frac{C}{m!}\int_{\mathbb{R}\setminus[\alpha,\beta]} \mathbb{E}|\mathcal{D}_r v_n(t)|^2 dr \leq C\epsilon.
\end{aligned}$$

By now it has been showed that $\{f_n^m(\cdot,t), n\in\mathbb{N}\}$ is relatively compact in $L^2(\mathbb{R}^m)$ for each finite m and fixed $t\in[a,b]$, which is equivalent to $\{v_n(t),\ n\in\mathbb{N}\}$ being pointwise relatively compact in $L^2(\Omega)$ for any fixed t.

But it is known from hypothesis (ii) that v_n are equi-continuous in time. So by generalized Arzelà-Ascoli Lemma, we conclude that $\{v_n\}_{n=1}^\infty$ is relatively compact in $C([a,b],L^2(\Omega))$. $\square$

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